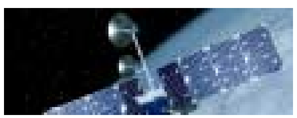
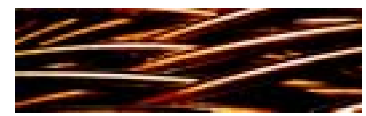
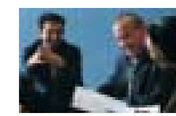
Techniques for Frequency Stability Analysis







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FREQUENCY CONTROL SYMPOSIUM

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Outline

- Introduction & Definitions
- Stability Analysis Overview
- Measurement Systems & Data Formats
- Preprocessing
- Stability Analysis
- Postprocessing & Reporting
- References

Introduction

- This tutorial describes practical techniques for time-domain frequency stability analysis.
- It covers the definitions of frequency stability, measuring systems and data formats, preprocessing steps, analysis tools and methods, postprocessing steps, and reporting suggestions.
- Examples are included for many of these techniques.
- Some of the examples use the Stable32 program [SW-6], which is a commercially-available tool for understanding and performing frequency stability analyses.
- Two good general references for this subject are NIST Technical Note 1337 [G-16] and a tutorial paper at the 1981 FCS [G-9].

- Note: The references are denoted by [X-#] where X is the topic code and # is the reference number.

Definitions

A frequency source has a sine wave output signal given by [ST-5]

$$V(t) = [V_0 + \varepsilon(t)]\sin[2\pi\upsilon_0 t + \varphi(t)]$$

where V_0 = nominal peak output voltage
$\varepsilon(t)$ = amplitude deviation
v_0 = nominal frequency
$\phi(t)$ = phase deviation

For the analysis of frequency stability, we are primarily concerned with the $\phi(t)$ term. The instantaneous frequency is the derivative of the total phase:

$$\upsilon(t) = \upsilon_0 + \frac{1}{2\pi}\frac{d\varphi}{dt}$$

For precision oscillators, we define the fractional frequency offset as

$$y(t) = \frac{\Delta f}{f} = \frac{v(t) - v_0}{v_0} = \frac{1}{2\pi v_0}\frac{d\varphi}{dt} = \frac{dx}{dt}$$

where

$$x(t) = \varphi(t) / 2\pi v_0$$

Stability Analysis

The time domain stability analysis of a frequency source is concerned with characterizing the variables *x(t)* and *y(t)*, the phase (expressed in units of time) and the fractional frequency, respectively. It is accomplished with an array of phase and frequency data arrays, x_i and y_i respectively, where the index i refers to data points equally-spaced in time. The x_i values have units of time in seconds, and the y_i values are (dimensionless) fractional frequency, $\Delta f/f$. The *x(t)* time fluctuations are related to the phase fluctuations by $\phi(t) = x(t)\cdot 2\pi\nu_0$, where ν_0 is the nominal carrier frequency in Hz. Both are commonly called "phase" to distinguish them from the independent time variable, *t*. The data sampling or measurement interval, τ_0, has units of seconds. The analysis or averaging time, τ, may be a multiple of τ_0 ($\tau = m\tau_0$, where m is the averaging factor).

The objective of a time domain stability analysis is a concise, yet complete, quantitative and standardized description of the phase and frequency of the source, including their nominal values, the fluctuations of those values, and their dependence on time and environmental conditions.

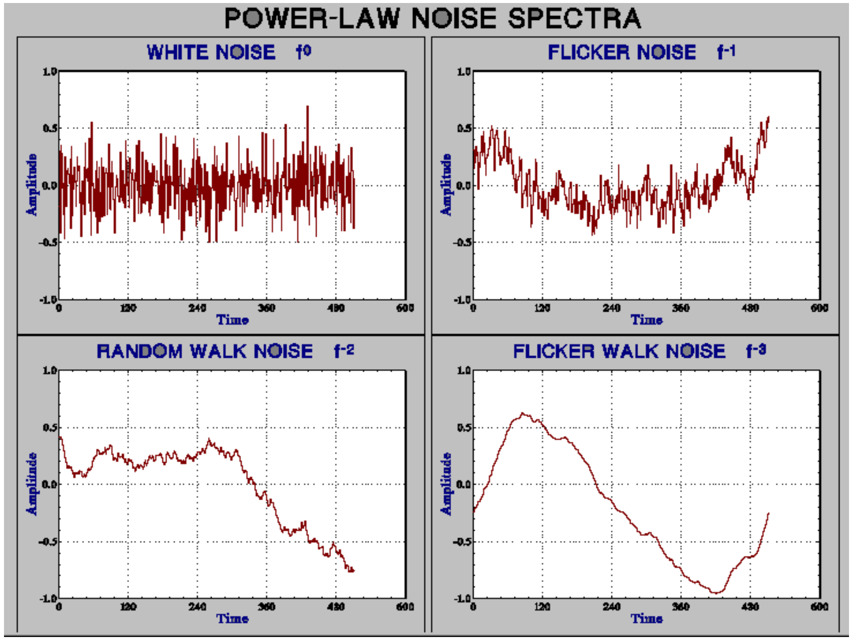
Stability Analysis (Con't)

A frequency stability analysis is normally performed on a single device, not a population of such devices. The output of the device is generally assumed to exist indefinitely before and after the particular data set was measured, which are the (finite) population under analysis. A stability analysis may be concerned with both the stochastic (noise) and deterministic properties of the device under test. It is also generally assumed that the stochastic characteristics of the device are constant (both stationary over time and ergodic over their population). The analysis may show that this is not true, in which case the data record may have to be partitioned to obtain meaningful results. It is often best to characterize and remove deterministic factors (e.g. frequency drift and temperature sensitivity) before analyzing the noise. Environmental effects are often best handled by eliminating them from the test conditions. It is also assumed that the frequency reference instability and instrumental effects are either negligible or removed from the data. A common problem for time domain frequency stability analysis is to produce results at the longest possible averaging times in order to minimize test time and cost. Analysis time is generally not as much of a factor.

Power-Law Clock Noise

A perfect frequency source would have a constant value equivalent to a single spectral line. It has been found that the instability of most frequency sources can be modeled by a combination of power-law noises having a spectral density of their frequency fluctuations of the form $S_y(f) \propto f^\alpha$, where f is the Fourier or sideband frequency in Hz.

Noise Type	Alpha
White PM	2
Flicker PM	1
White FM	0
Flicker FM	-1
Random Walk FM	-2



The even more divergent flicker walk (α=-3) and random run (α=-4) noise types are sometimes encountered.

Clock Noise (Con't)

The frequency stability analyst soon becomes familiar with these noise types, and the devices that display them. For example, passive atomic frequency standards have an inherent white FM noise characteristic that falls off with the square root of the averaging time until some flicker FM floor is reached (often caused by environmental effects). A summary of common frequency sources and their typical noises is shown below:

Source	Short Term	Medium Term	Long Term
Xtal Osc	W & F PM	F & RW FM	Aging
Rb Std	W FM	F FM	Aging
Cs Std	W FM	W FM	F FM
H Maser	W PM	W FM	RW FM & Aging
GPS Rx	W PM	Flywheel Osc	GPS System

Power Spectral Densities

The following power spectral densities are commonly used as frequency domain measures of frequency stability:

Formula	Units	Description
$S_y(f)$	1/Hz	PSD of fractional frequency fluctuations
$S_x(f)$	sec^2/Hz	PSD of time fluctuations
$S_\phi(f)$	rad^2/Hz	PSD of phase fluctuations
£(f)	dBc/Hz	SSB phase noise to carrier power ratio

where: PSD = Power Spectral Density
SSB = Single Sideband
dBc = Decibels with respect to carrier power

The relationship between these is:

$$S_x(f) = S_y(f)/(2\pi f)^2$$

$$S_\phi(f) = (2\pi\nu_o)^2 \cdot S_x(f)$$

$$£(f) = 10 \cdot \log[½ \cdot S_\phi(f)]$$

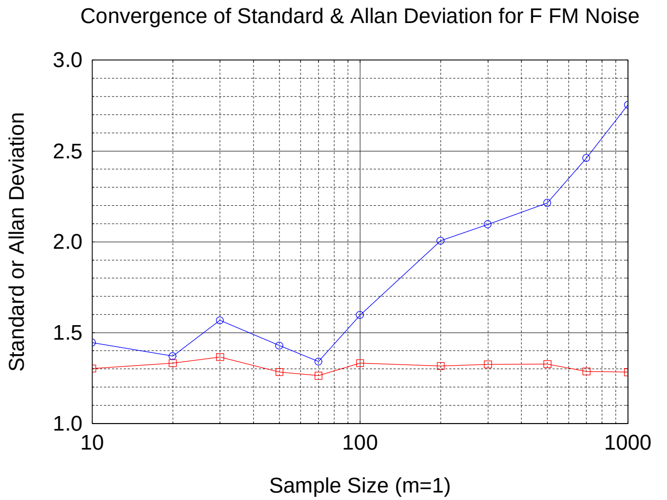
where ν_o is the carrier frequency, Hz.

Frequency Stability Statistics

- Statistical measures are used to characterize the fluctuations of a frequency source. These are 2nd-moment measures of scatter, much like the standard variance is used to quantify the variations in (say) the length of rods around a nominal value. The variations from the mean are squared, summed, and divided by the number of measurements -1.

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{y}\right)^2$$

- The result is often expressed as the square root, the standard deviation.
- Unfortunately, the standard variance does not converge to a single value for the non-white FM noises as the number of measurements is increased. Thus it is not a suitable statistic to describe the stability of most frequency sources.



Freq Stability Statistics (Con't)

- The Allan variance was developed to solve this problem. It uses 2nd differences of frequency (rather than differences from the mean) to calculate the variations, and is convergent for most clock noises.

$$\sigma_y^{\,2}(\tau) = \frac{1}{2(M-1)}\sum_{i=1}^{M-1}\left(y_{i+1} - y_i\right)^2 = \frac{1}{2(N-2)\tau^2}\sum_{i=1}^{N-2}\left(x_{i+2} - 2x_{i+1} + x_i\right)^2$$

- Other variances (e.g. Hadamard) have been devised that converge for all clock noises and handle frequency drift.

$$H\sigma_y^{\,2}(\tau) = \frac{1}{6(M-2)}\sum_{i=1}^{M-2}\left(y_{i+2} - 2y_{i+1} + y_i\right)^2 = \frac{1}{6(N-3)\tau^2}\sum_{i=1}^{N-3}\left(x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i\right)^2$$

- Still other variances provide PM noise discrimination (e.g. Modified Allan), or provide better confidence(e.g. Overlapping & Total).
- Thus the analyst has a number of effective statistical tools at his disposal to describe the instability of a frequency source.

Sigma-Tau Diagrams

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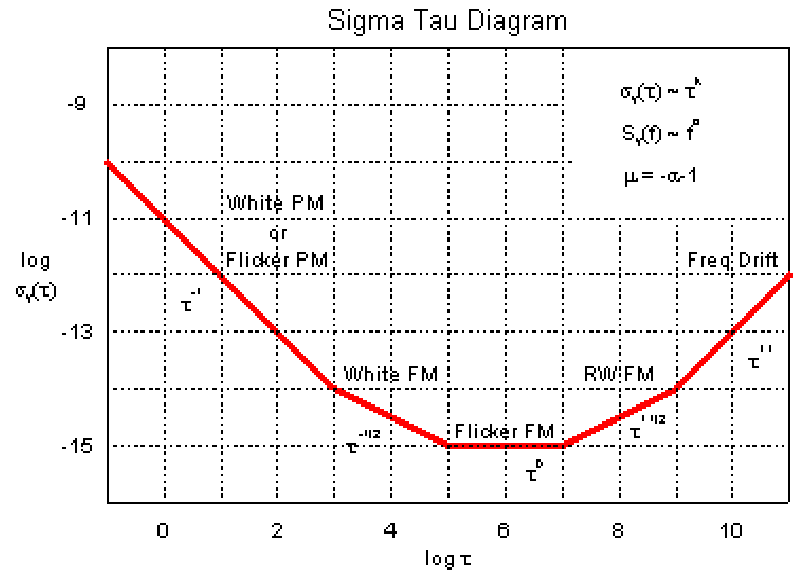
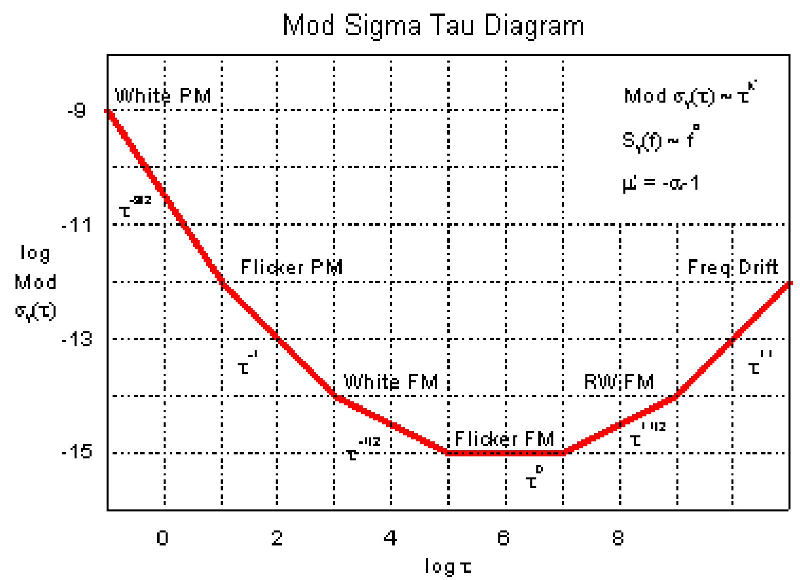
Sigma-Tau plots show the dependence of stability on averaging time, and are a common way to describe frequency stability. The power law noises have particular slopes, μ, on these log σ vs. log τ plots. α and μ are related as shown in the table below:

Noise	α	μ
W PM	2	-2
F PM	1	-2
W FM	0	-1
F FM	-1	0
RW FM	-2	1





Variance Types

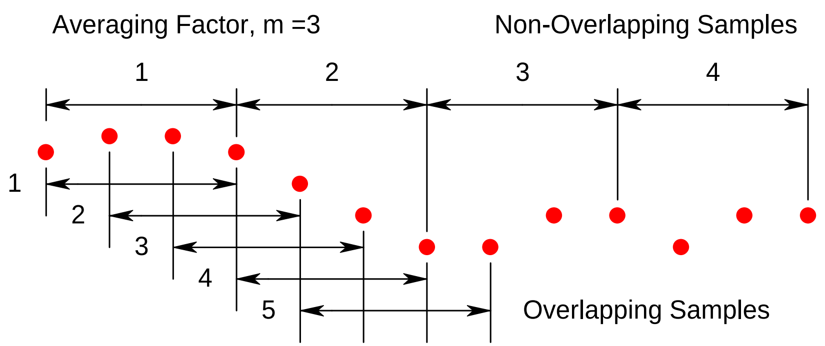
Variance Type	Characteristics
Standard	Non-convergent for some clock noises – Don't use
Allan	Classic – Use only if required – Poor confidence
Overlapping Allan	General Purpose - Most widely used – 1st choice
Modified Allan	Used to distinguish White and Flicker PM
Time	Based on modified Allan variance
Hadamard	Rejects frequency drift
Overlapping Hadamard	Better confidence than normal Hadamard
Total	Better confidence at long averages
Modified Total	Better confidence than modified Allan deviation
Time Total	Better confidence than for time deviation
Hadamard Total	Better confidence than for Hadamard deviation
Thêo1	Provides information over full record length

Variance Types (Con't)

- All are 2nd-moment measures of dispersion – scatter or instability of frequency from central value.
- All are usually expressed as deviations.
- All are normalized to standard variance for white FM noise.
- All except standard variance converge for common clock noises.
- Modified types have additional averaging that can distinguish W and F PM noises.
- Time variances based on modified types.
- Hadamard types also converge for FW and RR FM noise.
- Overlapping types provide better confidence than classic Allan variance
- Total types provide better confidence than overlapping.
- Thêo1 (Theoretical Variance #1) provides stability data out to nearly the full record length.
- Some are quite computationally-intensive, especially if results are wanted at all (or many) averaging times.

Fully Overlapping Samples

- Some stability calculations can utilize (fully) overlapping samples:



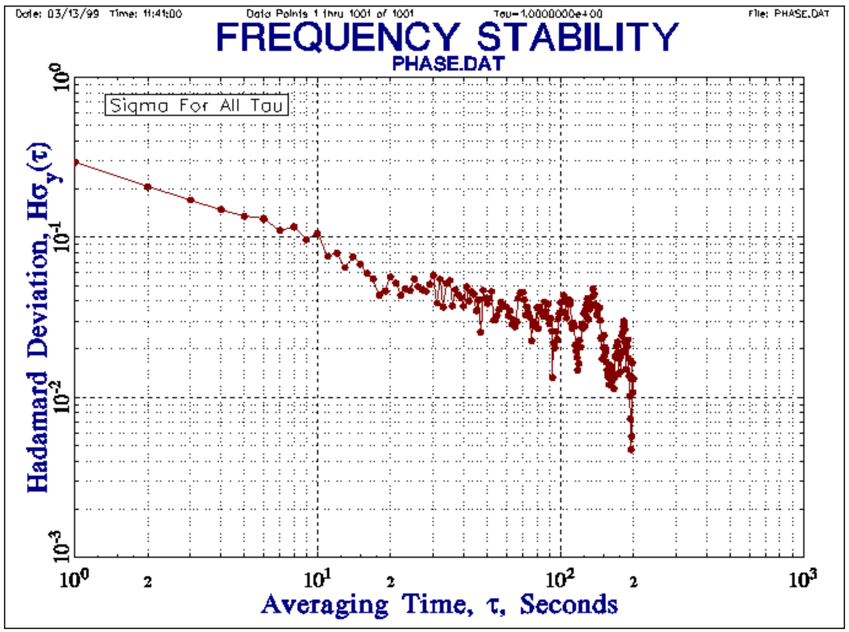
$$\sigma_y^2(\tau) = \frac{1}{2(M-1)} \sum_{i=1}^{M-1} \left(y_{i+1} - y_i \right)^2$$

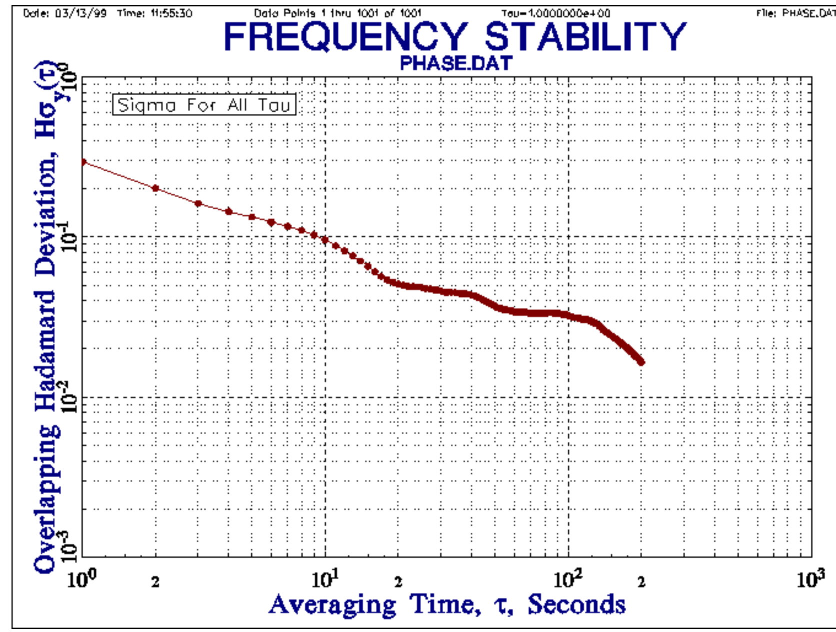
$$\sigma_y^2(\tau) = \frac{1}{2m^2(M-2m+1)} \sum_{j=1}^{M-2m+1} \sum_{i=j}^{j+m-1} \left(y_{i+m} - y_i \right)^2$$

- The use of overlapping samples improves the confidence of the resulting stability estimate at the expense of greater computational time.
- The overlapping samples are not completely independent but nevertheless do increase the effective number of degrees of freedom (see later) and thereby improve the confidence in the results.
- The choice of overlapping samples applies to the Allan and Hadamard variances. Other variances (e.g. total) always use them.
- Overlapping samples don't apply at the basic measurement interval, which should be as short as practical to support a large number of overlaps at longer averaging times.

Overlapping Samples (Con't)

The following plots show the significant improvement in statistical confidence obtained by using overlapping samples in the calculation of the Hadamard deviation:





$$H\sigma_y^2(\tau) = \frac{1}{6(M-2)} \sum_{i=1}^{M-2} \left(y_{i+2} - 2y_{i+1} + y_i\right)^2$$

$$H\sigma_y^2(\tau) = \frac{1}{6m^2(M-3m+1)} \sum_{j=1}^{M-3m+1} \sum_{i=j}^{j+m-1} \left(y_{i+2m} - 2y_{i+m} + y_i\right)^2$$

MDEV to Identify W & F PM Noise

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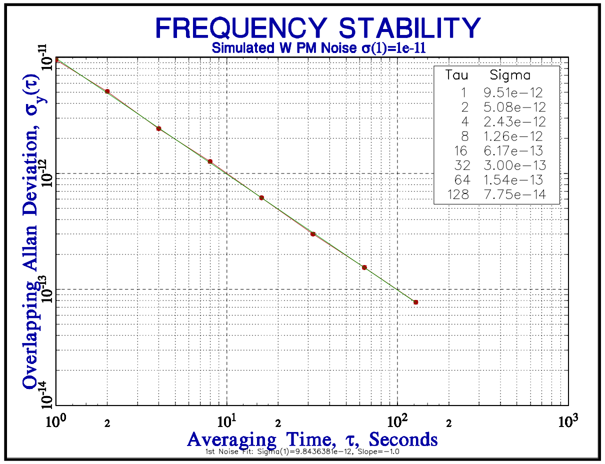
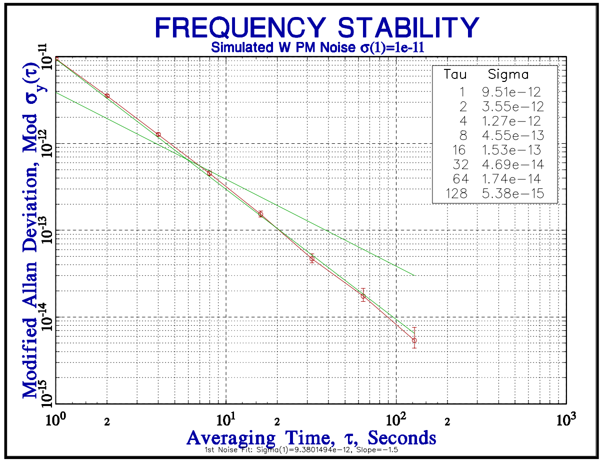
ADEV

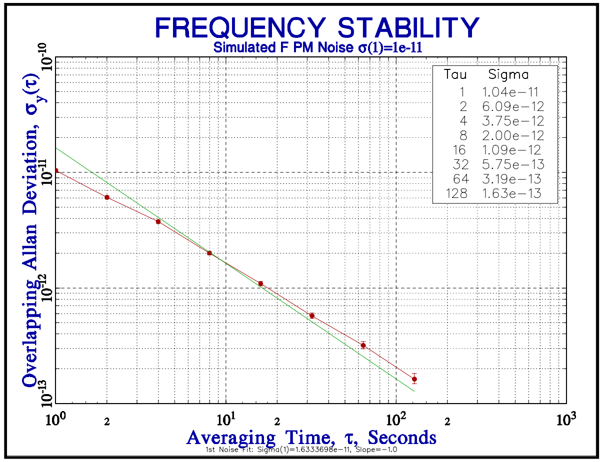
MDEV

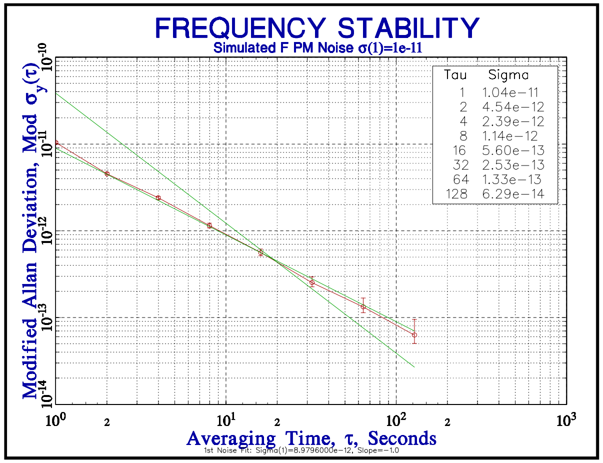
W PM





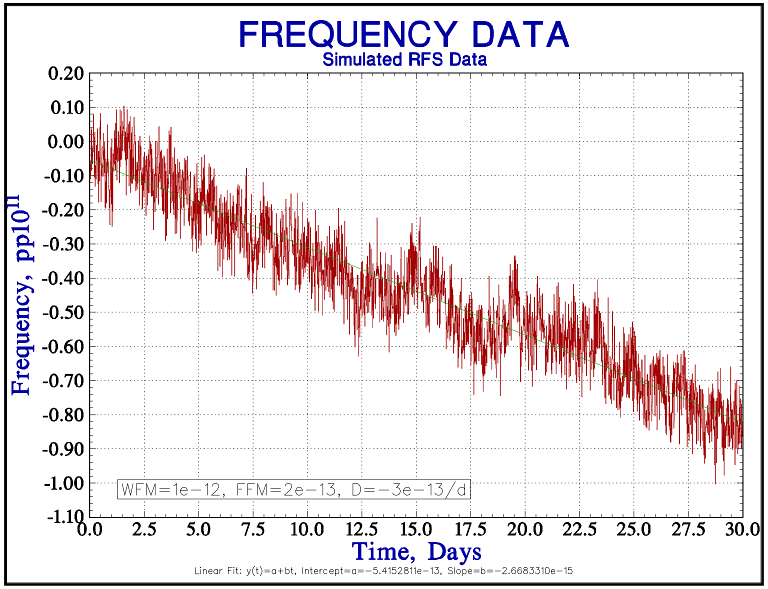
F PM

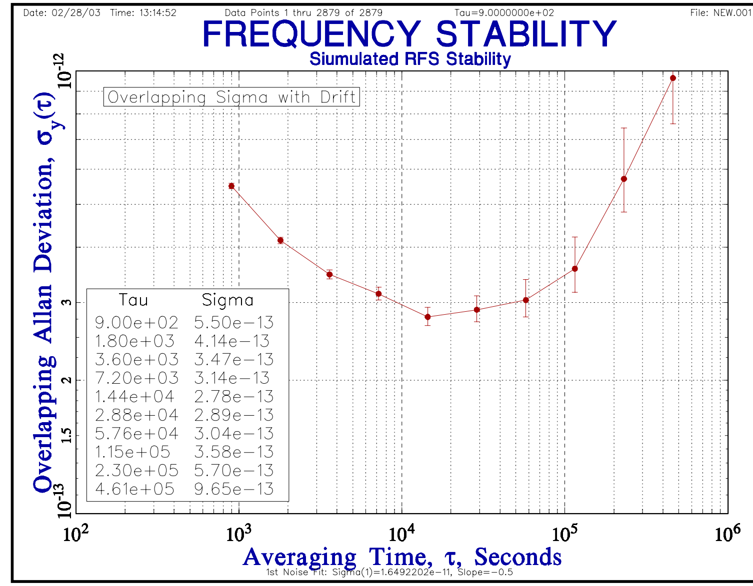


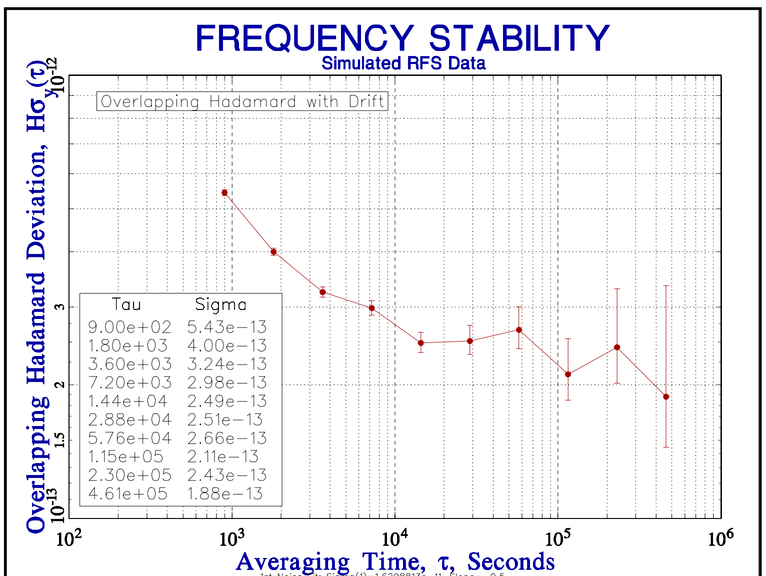


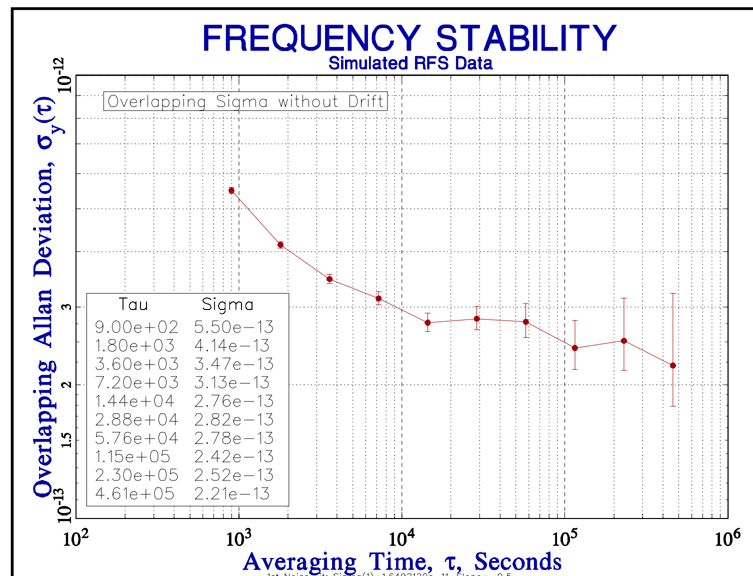
The W and F FM noise slopes are both ≈ -1.0 on the ADEV plots, but they can be distinguished as –1.5 and –1.0 respectively on the MDEV plots.

Hadamard Deviation to Reject Drift







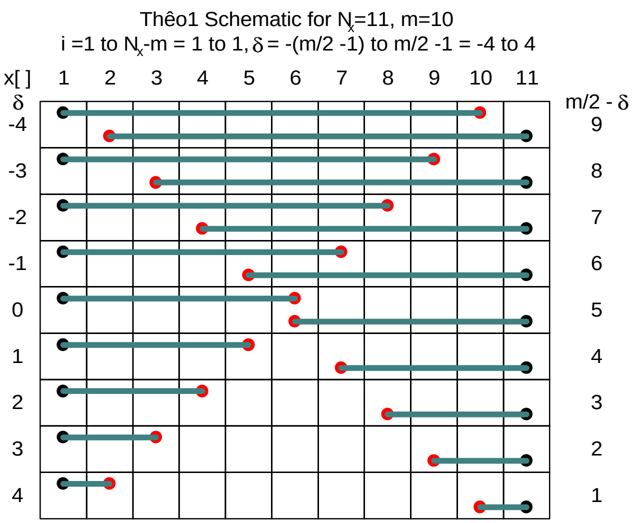


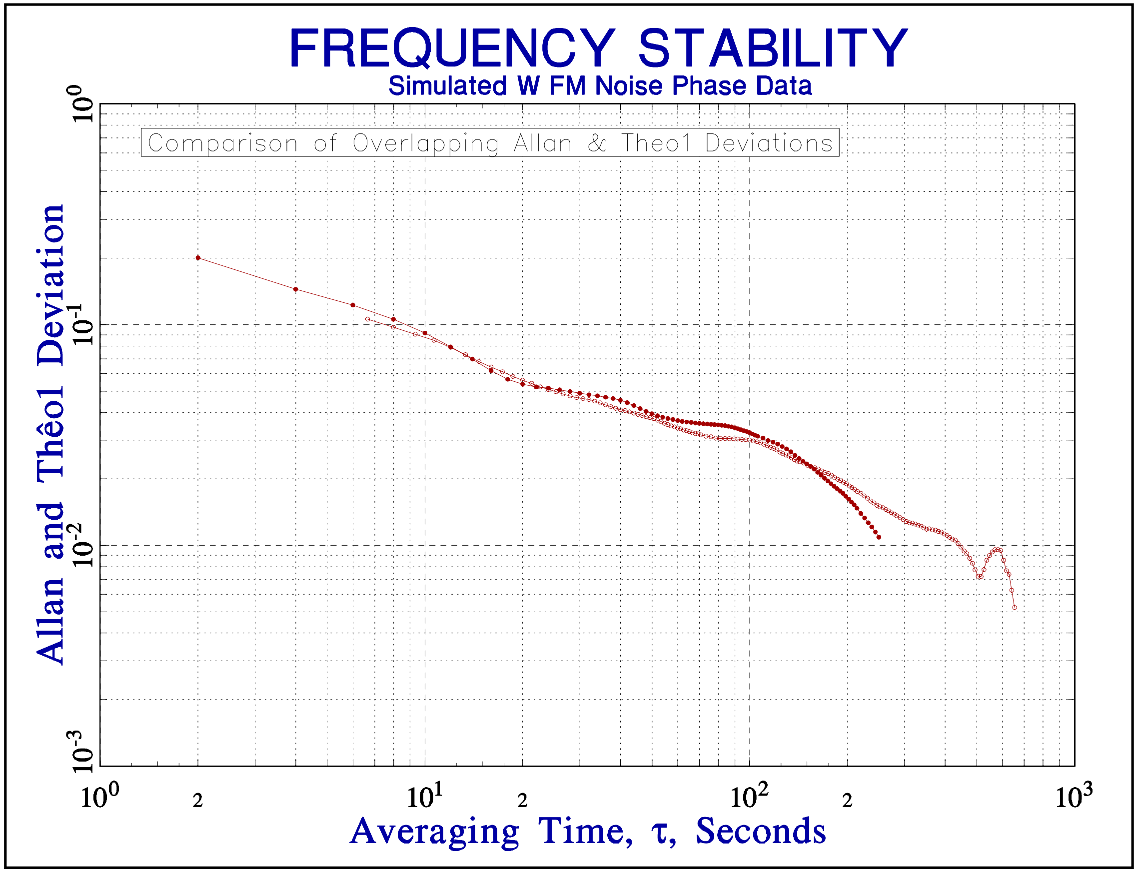
Thêo1

Thêo1 [TH-1] is a new statistic currently under development that offers the ability to provide stability data at larger averaging factors.

$$Th\hat{e}o1(m-1,\tau_0,N_x)=\frac{1}{(N_x-m)(m\tau_0)^2}\sum_{i=1}^{N_x-m}\sum_{\delta=-(m/2-1)}^{(m/2-1)}\frac{1}{(m/2-\delta)}\left[(x_i-x_{i-\delta+m/2})+(x_{i+m}-x_{i+\delta+m/2})\right]^2$$

Thêo1 provides useful samples at an averaging factor, m, nearly equal to the record length N_x ($m_{max}=N_x-1$)





MTIE and TIE rms

- The maximum time interval error (MTIE) and rms time interval error (TIE rms) are clock stability measures commonly used in the telecom industry [M-3], [M-5].
- MTIE is determined by the extreme time deviations within a sliding window of span τ. It is not as easily related to clock noise processes as TDEV [M-1].
- MTIE is computationally-intensive for large data sets [M-7].
- For no frequency offset, TIE rms is approximately equal to the standard deviation of the fractional frequency fluctuations multiplied by the averaging time. It is therefore similar in behavior to TDEV, although the latter is better suited for divergent noise types.

Overall Stability Analysis System

- Unit Under Test and Supporting Equipment
  - Power, Monitoring & Environmental Control
- Reference Standard and Calibration Equipment
  - Preferably More Stable than UUT
- Clock Measuring System
  - Phase Data Preferred, High Resolution Required, No Dead Time Preferred
- Data Acquisition and Storage
  - Time-Tagging, Data Formats, Multiple Channels, Server
- Analysis Workstation
  - Offline from Measuring System
- Analysis Software
  - Specialized Stability Statistics
- Analysis Techniques
  - Outlier and Drift Removal, Noise Recognition, Allan and Other Variances
- Reporting Tools
  - Plotting, Model Fitting, Interpretation of Results

Measurement Systems

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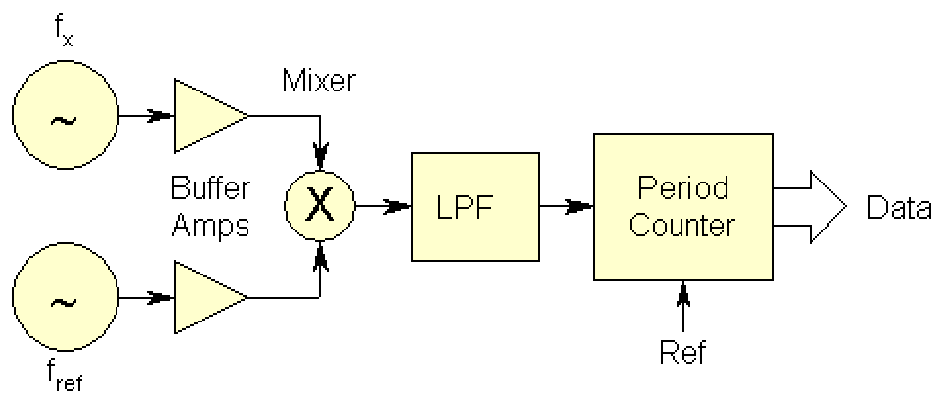
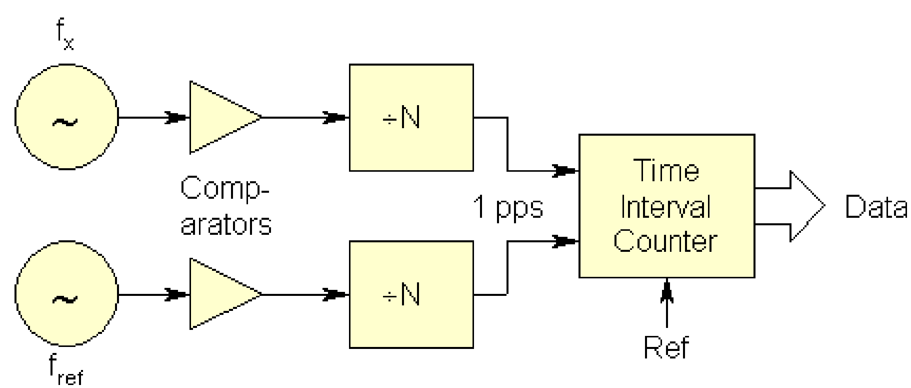
- A frequency counter can be used to directly make frequency measurements with modest resolution
- Higher resolution can be obtained from the following popular measurement system configurations:

- Heterodyne System
  - Offset Reference
  - Narrowband
  - High Resolution
  - Frequency Only
  - Dead Time

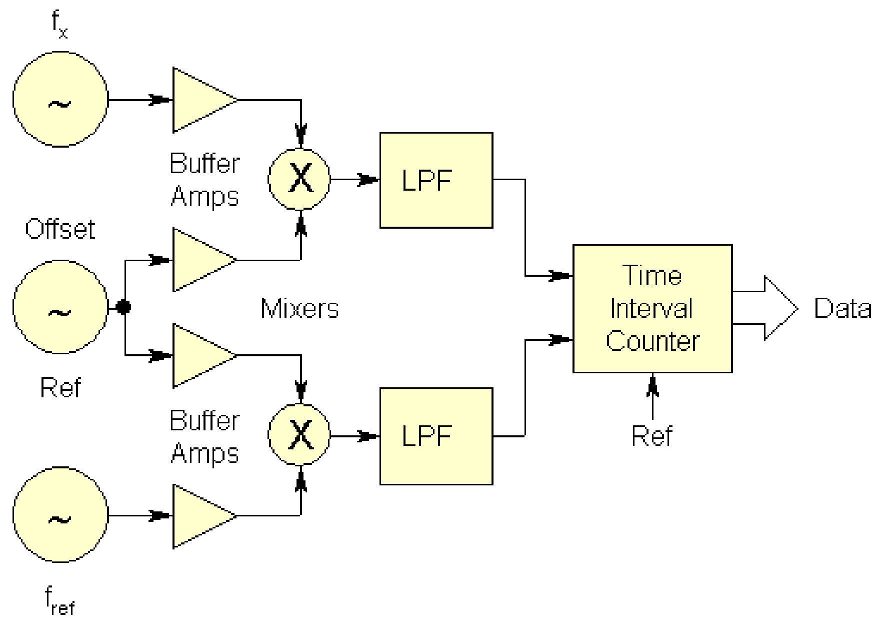


- Time Interval Counter
- Std Freq Reference
  - Wide Freq Range
  - Fair Resolution
  - Phase Measured
  - No Dead Time
  - Simple



Measurement Systems (Con't)

- Dual Mixer Time Interval
  - Offset Reference
  - Narrowband
  - High Resolution
  - Phase Measured
  - No Dead Time
  - Multiple Channels
  - Complex



- High resolution clock measuring systems are available commercially [ME-6], [ME-7], [ME-8]

Data Format

- The essential data are an array of equally-spaced phase or frequency values taken at particular measurement interval.
- The data must have sufficient resolution to support the analysis. This can require many orders of magnitude of dynamic range, particularly for frequency data taken with an ordinary frequency counter, or long-term phase data for a source having a large frequency offset.
- The data may have an associated time tag array, which can be a significant advantage for relating the data to other events. Use of the Modified Julian Date (MJD) is recommended.
- Data is most often stored as numeric ASCII characters, which provides the easiest transfer between different hardware and software.
- Non-numeric characters are often included as header information or comments, but are best put on separate lines.

Phase vs. Frequency Data

- Phase data is preferred because it can be used to obtain frequency data, but the reverse it not strictly true. Absolute phase cannot be reconstructed from frequency data, and a gap in the frequency data means that phase continuity is lost.
- Phase data can be used at a longer averaging time by simply re-sampling (decimating) it. Frequency data must be averaged to accomplish this, a process that takes much longer in a stability algorithm. It is generally faster (sometimes much faster) to convert frequency data to phase data before performing a stability calculation.
- Phase data applies directly to timing applications, and is fundamental for time distribution systems.
- Frequency data is often easier to “read”. Outliers are more apparent, frequency changes are directly seen, and drift is more obvious.
- Frequency data is the more fundamental for most internal aspects of a frequency source.

Basic Analysis Sequence

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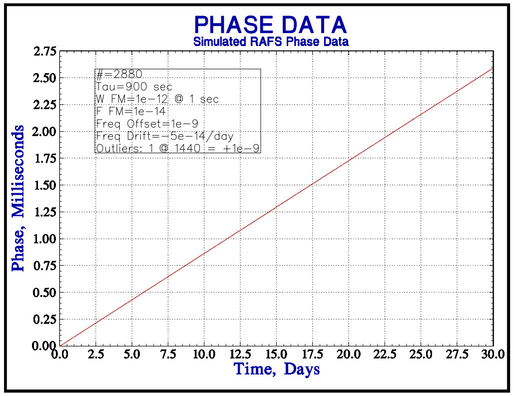
Phase Data



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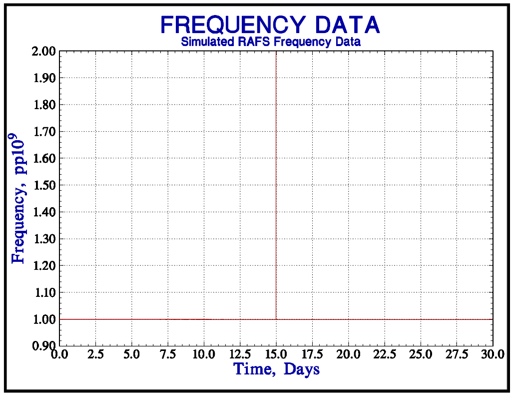
Convert to Freq



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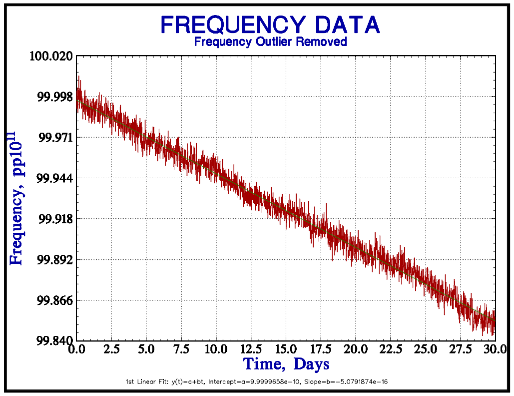
Remove Outlier

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0.000000000000000e+00
8.999873025741449e-07
1.799890526185009e-06
2.699869215098003e-06
3.599873851209537e-06
4.499887627663997e-06
5.399836191440859e-06
6.299833612216789e-06
7.199836723454638e-06
8.099785679257264e-06
8.999774896024524e-06
9.899732698242008e-06
Etc.
```

Outlier obvious in freq data – must remove it to continue analysis.

Can now see noise and drift.

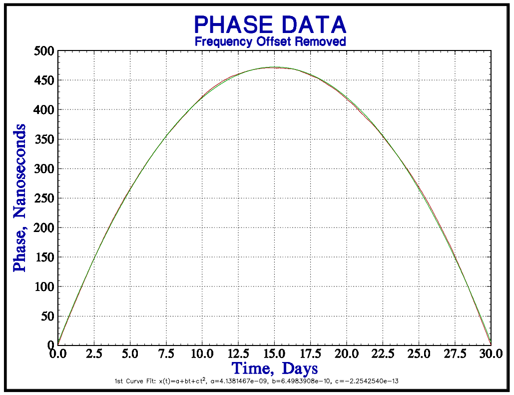
Phase data is just a ramp with slope corresponding to freq offset.

Visual inspection of data is an important preprocessing step!

Basic Analysis Sequence (Con't)

Remove Freq Offset

→



Remove Freq Drift

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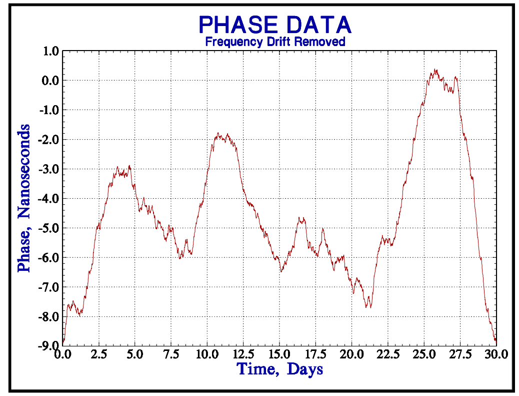

Get Freq Residuals

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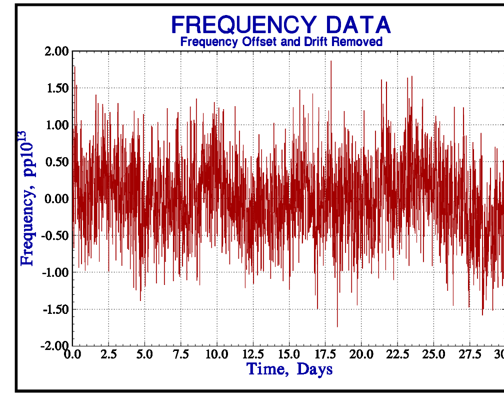


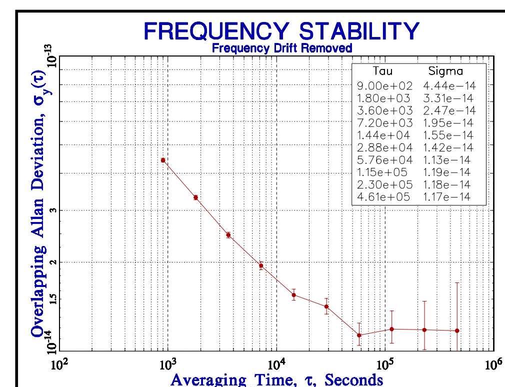
Quadratic shape is due to freq drift - can now begin to see phase fluctuations.

Phase and freq residuals allow noise to be clearly seen.

Notice W & F FM Slopes at simulated levels

↓

Analyze Stability



Preprocessing

- Preprocessing is usually necessary before the formal stability analysis is started.
- Phase data may be decimated or frequency data averaged to a longer tau. This will shorten the data file and make subsequent processing faster, but is a disadvantage for overlapping and total statistics.
- Phase <-> Frequency conversions may be performed.
- The data may be plotted and examined visually for steps, jumps, spikes, glitches, interference and other anomalies.
- A portion of the data may be selected for analysis.
- Outliers may be found and removed (with discretion). One should understand why the data point is invalid.
- Frequency offset may be removed.
- Frequency drift may be analyzed and removed.

Phase-Frequency Conversions

- Phase data can be converted to frequency data by dividing the 1st difference of phase by tau: $y_i = (x_i - x_{i-1}) / \tau$.
- Frequency data can be converted to phase data by multiplying the frequency by tau and adding it to the phase: $x_i = x_{i-1} + (y_{i-1}) \cdot \tau$.
- Phase to frequency conversion is straightforward unless the two phase values are the same, which gives y = 0, and can be confused with a gap. Using a small non-zero value (e.g. 1e-99) will solve this. But two identical adjacent phase values can be a sign of data quantization or a measurement problem.
- Frequency to phase conversion is undefined if there is a gap in the data. To preserve phase continuity, the average frequency value can be used to integrate through the gap.
- The individual time tag interval can be used instead of a fixed tau for non uniformly-spaced data.

Outliers

- It is important to have (and use) a consistent way to identify and remove outliers, based on the the methods of robust statistics [R-2].
- This is much easier to do for frequency (rather than phase) data.
- An outlier is an extreme data point that is significantly larger or smaller than most others (often a case of "I'll know it when I see it").
- The median is a robust way to determine the center value (an outlier affects the mean).
- The deviation from the median is a robust way to determine whether a point is an outlier.
- The median absolute deviation (MAD) [R-3] is a robust way to set the criterion for an outlier. It is the median of the (scaled) absolute deviations of the data points from their median value, and is defined as:

  $$\mathrm{MAD} = \mathrm{Median}\{\ |y(i) - m| / 0.6745\ \}$$

  where $m=\mathrm{Median}\{ y(i) \}$, and the factor 0.6745 makes the MAD equal to the standard deviation for normally distributed data.
- An outlier criterion of 5X the MAD is usually a good choice.

Frequency Offset

Frequency offset may be calculated for phase data by either of two methods:

1. Linear Fit: The first (optimal for white PM noise) uses a least-squares linear fit to the phase data, x(t) = a + bt, where slope = y(t) = b.
2. Endpoints: The second method simply uses the difference between the first and last points of the phase data, slope = y(t) = [x(end) – x(start)] / (M-1), where M = # phase data points. This method (optimal for white FM noise) can be used to match the two endpoints.

Frequency offset is usually calculated for frequency data as the average (mean) of the frequency values.

Drift Analysis and Removal

Three methods are commonly used to analyze frequency drift in phase data:

1. Quadratic Fit: The first is a least-squares quadratic fit to the phase data, $x(t) = a + bt + ct^2$, where $y(t) = x'(t) = b + 2ct$, slope $= y'(t) = 2c$. This method is optimum for white PM noise [G-16].

2. 2nd Differences: The second method is the average of the 2nd differences of the phase data, $y(t) = [x(t+\tau)–x(t)]/\tau$, slope $= [y(t+\tau)–y(t)]/\tau = [x(t+2\tau)–2x(t+\tau)+x(t)]/\tau^2$.This method is optimum for random walk FM noise [D-4].

3. 3-Point: The third method uses the 3 points at the start, middle and end of the phase data, slope $= 4[x(end)–2x(mid)+x(start)]/(M\tau)^2$, where M = # data points It is the equivalent of the bisection method for frequency data [D-3].

Drift Analysis and Removal (Con't)

Symmetricom

Four methods are commonly used to analyze frequency drift in frequency data:

1. Linear Fit: The first, the default, is a least squares linear regression to the frequency data, $y(t) = a+bt$, where a = intercept, b = slope = $y'(t)$. This is the optimum method for white FM noise.
2. Bisection: The second method computes the drift from the frequency averages over the first and last halves of the data, slope = 2 [y(2nd half) – y(1st half)] / ($N\tau$), where N = # points. This bisection method is optimum for white and random walk FM noise.
3. Log Fit: The third method, a log model of the form (see MIL-O-55310B), $y(t) = a \cdot \ln(bt+1)$, where slope = $y'(t) = ab/(bt+1)$ which applies to frequency stabilization.
4. Diffusion Fit: The last frequency drift method is a diffusion ($\sqrt{\ }$ t) model of the form $y(t) = a+b(t+c)^{1/2}$, where slope=$y'(t)=½\cdot b(t+c)^{-1/2}$.

Bias and Confidence

Symmetricom

- It is not enough to simply calculate one of these statistics. It is also necessary to know what value it is expected to converge to, and how closely it has done so.
- The Allan variance is defined so that its expected value is the same as the standard variance for white FM noise. Most of the other clock statistics are either defined likewise, or are biased estimators of the Allan variance for which (noise and sample size dependent) corrections must be applied.
- Thus, properly applied and corrected, all of these statistics converge to the same expected value, with increasing confidence as the number of samples increases.
- The uncertainly in the result can be estimated and used to set confidence intervals and error bars, using the theory of X^2 statistics that applies to variances.
- This requires knowledge of the equivalent number of degrees of freedom (EDF), which is also a function of the noise type and number of samples.

Noise Identification

- Noise type identification is important not only for understanding the physical basis of the instability but also to apply bias corrections and to set confidence limits.
- The noise type may be known a priori, or can be determined by a preliminary analysis from the slope of a log σ vs. log τ stability plot.
- Preferably, the noise identification is performed automatically at each averaging time during a stability run to support bias corrections and error bar determination.
- A way for automatic noise identification is to calculate additional variances (e.g. standard or mod Allan) whose ratio to the desired variance (e.g. Allan) is dependent in a known way on the noise type [N-6].
- The Barnes B_1 and R(n) ratios can be used for this purpose.
- B_1 is the ratio of the standard (N-sample) and Allan (2-sample) variances [N-3].
- R(n) is the ratio of the modified Allan and Allan variances [G-16].
- Analytic expressions are available to relate these ratios to the noise type and number of data samples.

Noise Identification (Con't)

The dominant power law noise type can be estimated by comparing the ratio of the N-sample (standard) variance to the 2-sample (Allan) variance of the data (the B_1 bias factor) to the value expected of this ratio for the pure noise types (for the same averaging factor). This method of noise identification, while not perfect, is reasonably effective in most cases. The main limitations are (1) its inability to distinguish between white and flicker PM noise, and (2) its limited precision at large averaging factors where there are few analysis points. The former limitation can be overcome by supplementing the B_1 ratio test with one based on R(n), the ratio of the modified Allan variance to the normal Allan variance. That technique is applied to members of the modified family of variances (MVAR, TVAR, and MTOT). The second limitation can be avoided by using the previous noise type estimate at the longest averaging time of an analysis run. One further limitation is that the R(n) ratio is not meaningful at a unity averaging factor. A description of this power law noise identification method can be found in Reference [N-6].

Confidence Intervals

Sample variances are distributed according to the expression:

$$\chi^2 = \frac{EDF \cdot s^2}{\sigma^2}$$

where χ^2 is the Chi-square, s^2 is the sample variance, σ^2 is the true variance, and EDF is the Equivalent number of Degrees of Freedom (not necessarily an integer). The EDF is determined by the number of analysis points and the noise type. Single or double-sided confidence intervals (error bars) with a certain confidence factor may be set for variances based on χ^2 statistics. The general procedure is to choose a confidence factor, p, calculate the corresponding χ^2 value, determine the EDF from the variance type, noise type and number of analysis points, and then set the statistical limit(s) on the variance. For double-sided limits:

$$\sigma^2_{min} = s^2 \cdot \frac{EDF}{\chi^2(p, EDF)} \quad \text{and} \quad \sigma^2_{max} = s^2 \cdot \frac{EDF}{\chi^2(1-p, EDF)}$$

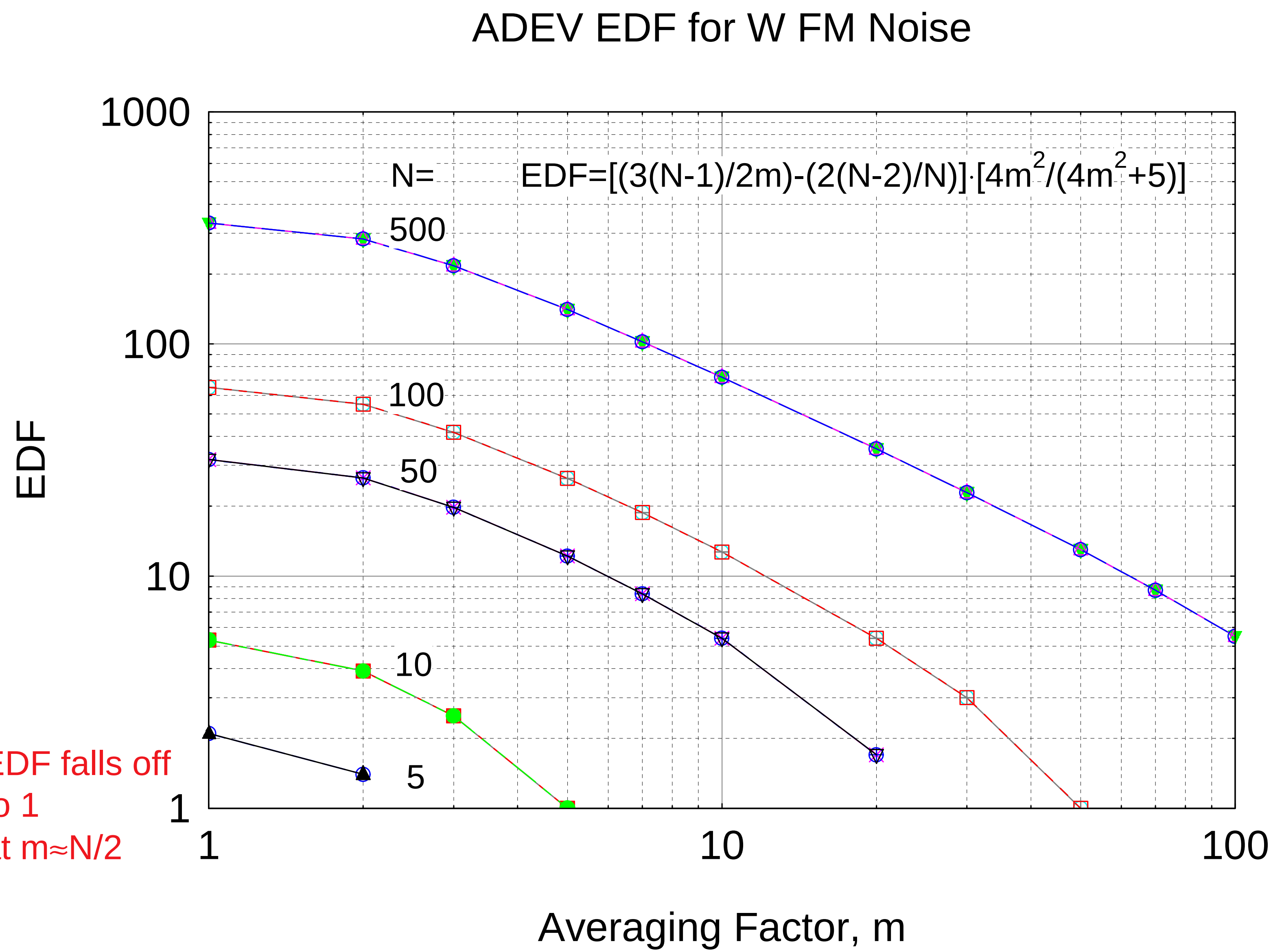
EDF

- The Equivalent # of χ^2 Degrees of Freedom (EDF) is needed to determine confidence intervals and set error bars for a variance estimate.
- The EDF value depends on the variance type, the noise type, and the number of data points used in the analysis.
- Empirical formulae exist for determining the approximate EDF value.
- Some variance types use a large number of highly-correlated 2nd differences to obtain a larger EDF for better confidence.

Variance Type	EDF Determination Method	Reference
Allan & Hadamard	$\sigma/\sqrt{N}$ & K_n, see p. TN-182	[G-16]
Overlapping Allan	See Table 12.4 (as corrected)	[G-11]
Modified & Time Allan	HEDF w/ modified args	[HV-8]
Overlapping Hadamard	HEDF	[HV-8]
Total	See Table I	[T-7]
Modified & Time Total	See §4.2 & Table 1	[MT-1]
Hadamard Total	See Eq. (7) & Table 1	[HV-9]
Thêo1	Currently Under Investigation	[TH-1]

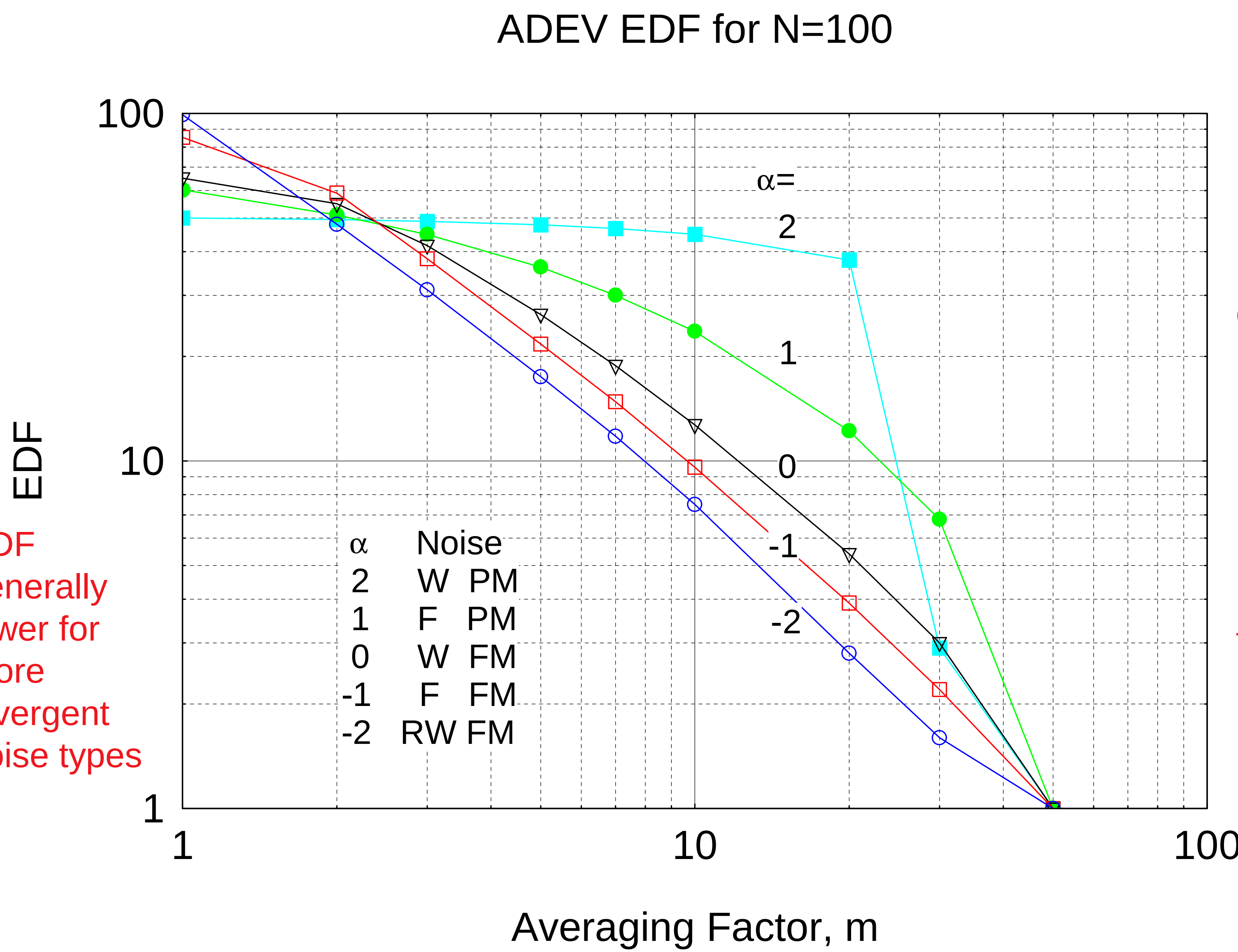
ADEV EDF Example

Symmetricom



EDF falls off to 1 at m≈N/2

ADEV EDF Example (Con't)

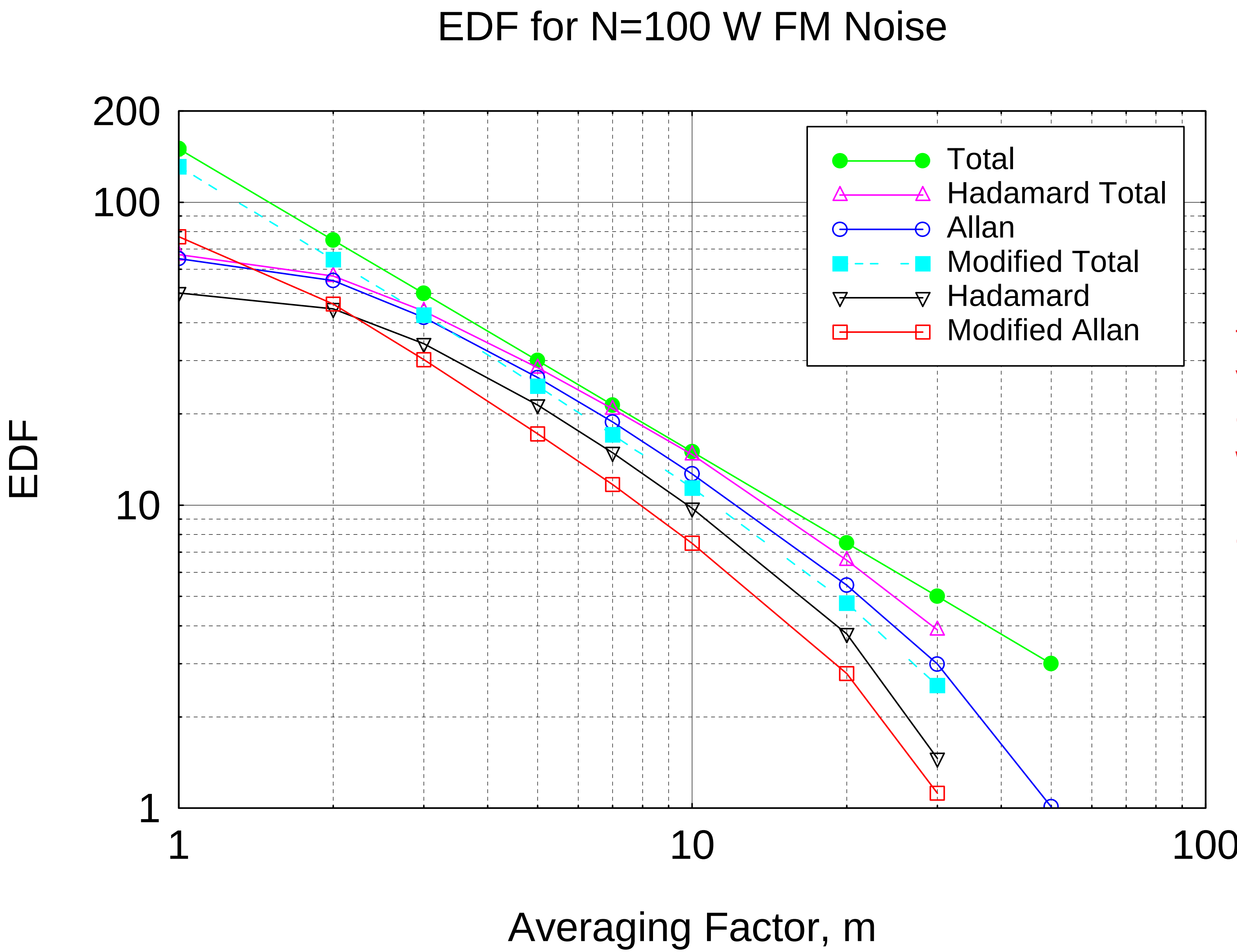


EDF generally lower for more divergent noise types

EDF nearly independent of m for W PM noise

EDF ≈ N/m for $\alpha \leq 0$

EDF vs Variance Type



EDF larger for total variances compared with their non-total equivalents

Postprocessing

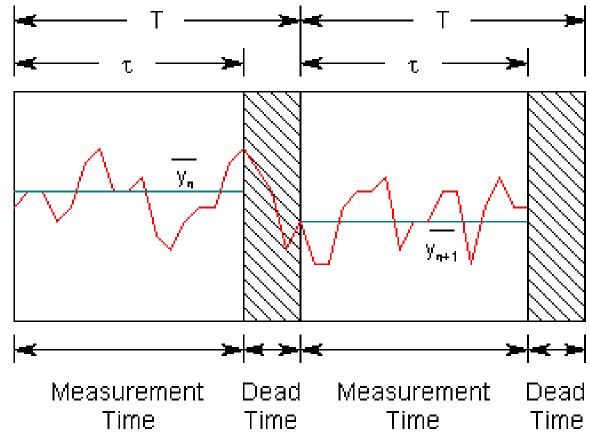
- Dead Time Correction
  - Apply Barnes B_2 and B_3 Bias Ratio
  - Important for non-white noises and large dead time ratios
- Reference Correction
  - Separation of Variances
- 1 of 2 Clocks Correction
  - Divide sigma by $\sqrt{2}$
  - Use when measuring 2 identical units
- Noise Line Fitting
  - Fit power law noise lines to stability data
- Interpretation of Results
  - Data presentation and discussion
  - Correlation with clock physics
  - Understanding of environmental effects
  - Explanations for anomalies
  - Suitability for application

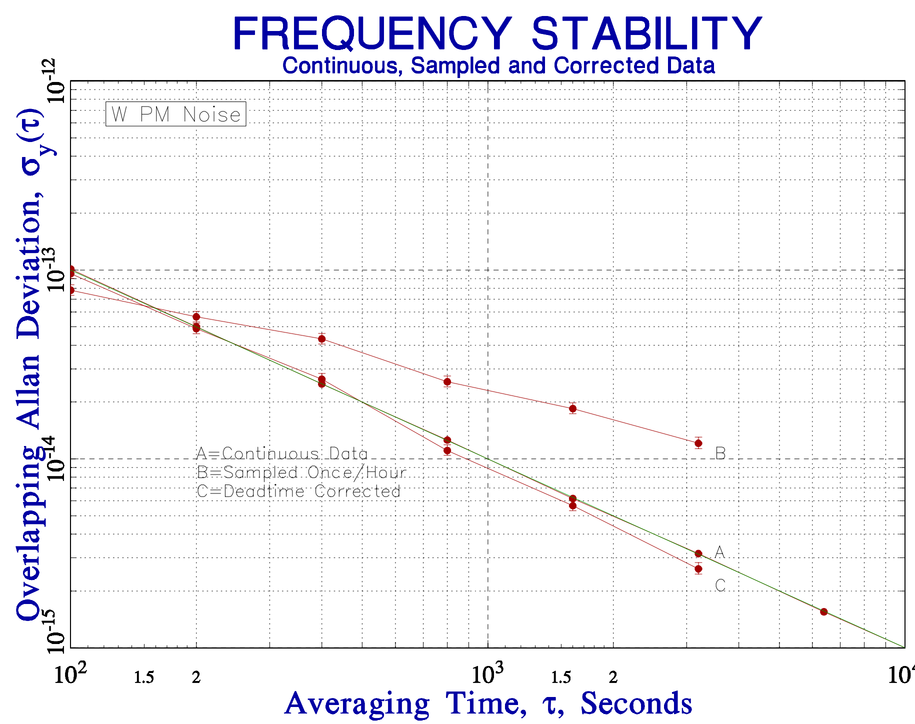
Dead Time

Dead time between frequency measurements can affect the results of a stability analysis, and should be avoided if possible.

Otherwise, the Barnes B_2 and B_3 bias ratios should be used to correct for the effect of dead time [DT-2].

This is an example of the effect of dead time on W PM noise sampled with a 100-second measurement once per hour, and the use of dead time corrections.





Separation of Variances

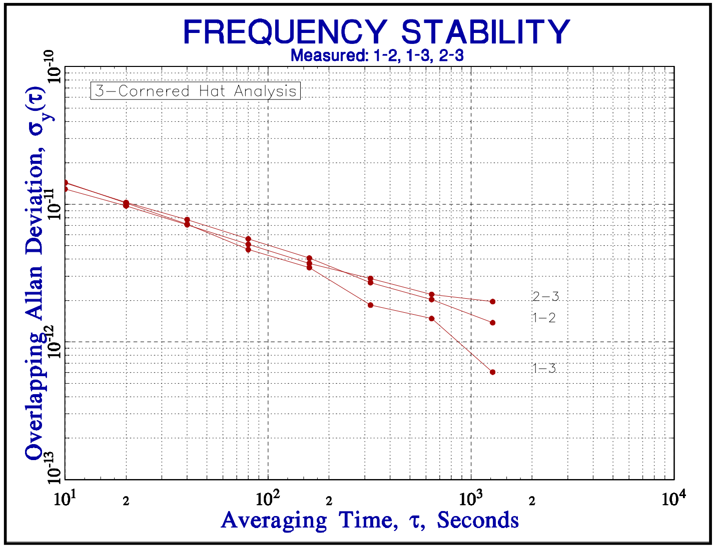
- The so-called "3-Cornered Hat" method can be used to separate the variances of the unknown and reference clocks in a stability measurement [3C-2], [3C-10].
- Three measurements are made of the sources in pairs, and are processed to obtain the individual variances.

$$\sigma_A^2 = \tfrac{1}{2}\left(\sigma_{AB}^2 + \sigma_{AC}^2 - \sigma_{BC}^2\right)$$

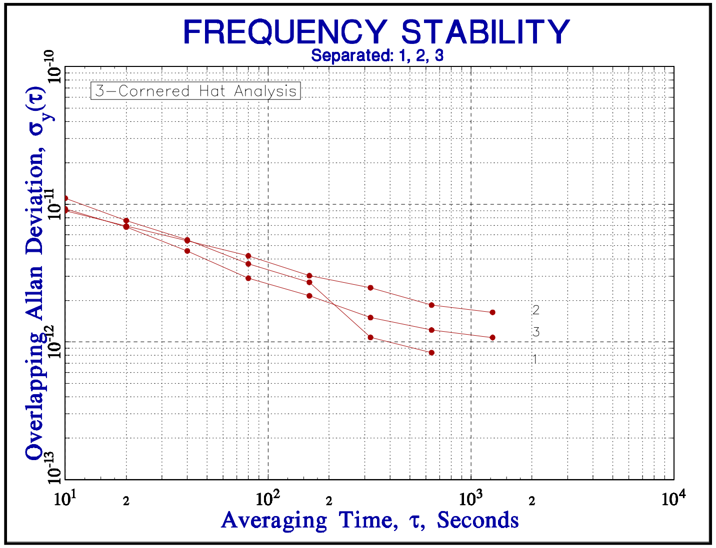
$$\sigma_B^2 = \tfrac{1}{2}\left(\sigma_{AB}^2 + \sigma_{BC}^2 - \sigma_{AC}^2\right)$$

$$\sigma_C^2 = \tfrac{1}{2}\left(\sigma_{AC}^2 + \sigma_{BC}^2 - \sigma_{AB}^2\right)$$

- The method works best with large data sets for uncorrelated devices having similar stabilities – otherwise non-physical negative variances can result. A "perfect" reference is still the best!



Measured ADEVs

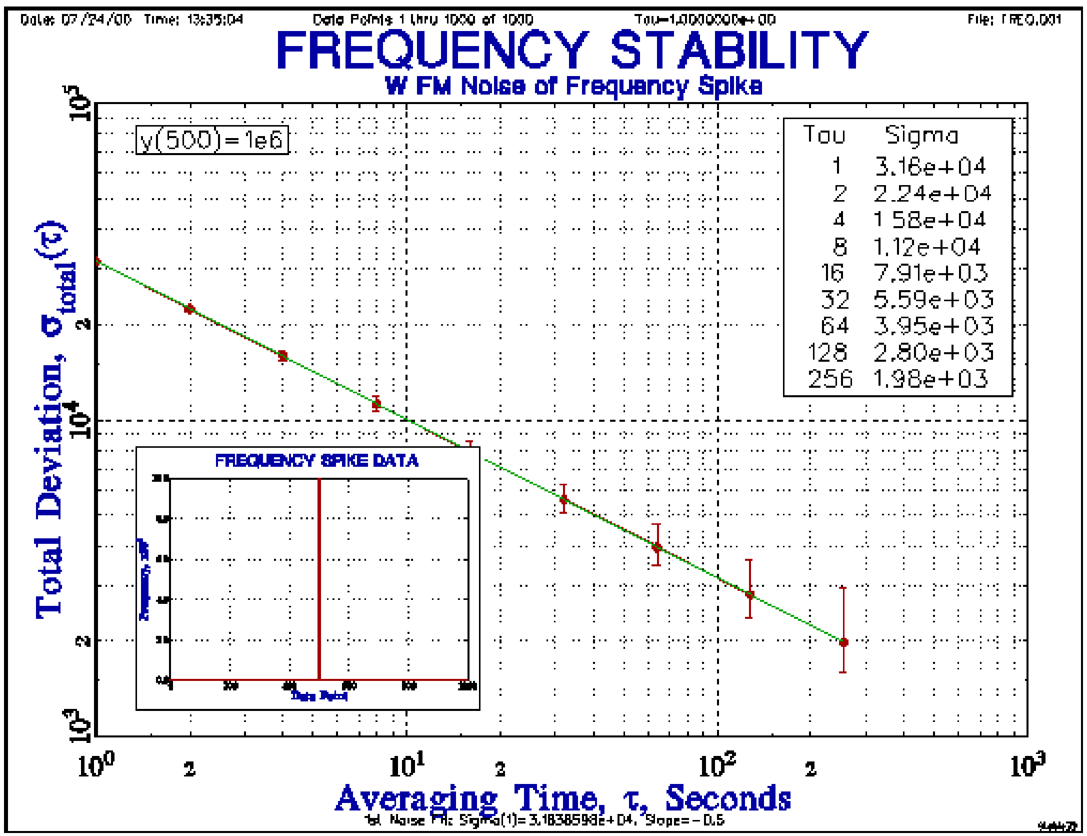


Separated ADEVs

Phase Steps/Frequency Spikes

Symmetricom

The Allan deviation of a frequency record having large spike (a phase step) has a $\tau^{-1/2}$ characteristic [G-13]. This can be a source of confusion, but can also be used as a means for simulation. For example, adding a single large central outlier (e.g. 10^{6}) to an otherwise much smaller 1000-point data set yields a white FM noise level of $\sigma_y(\tau_0)=[(10^6)^2/(1000-1)]^{1/2} = 3.16386e4$.

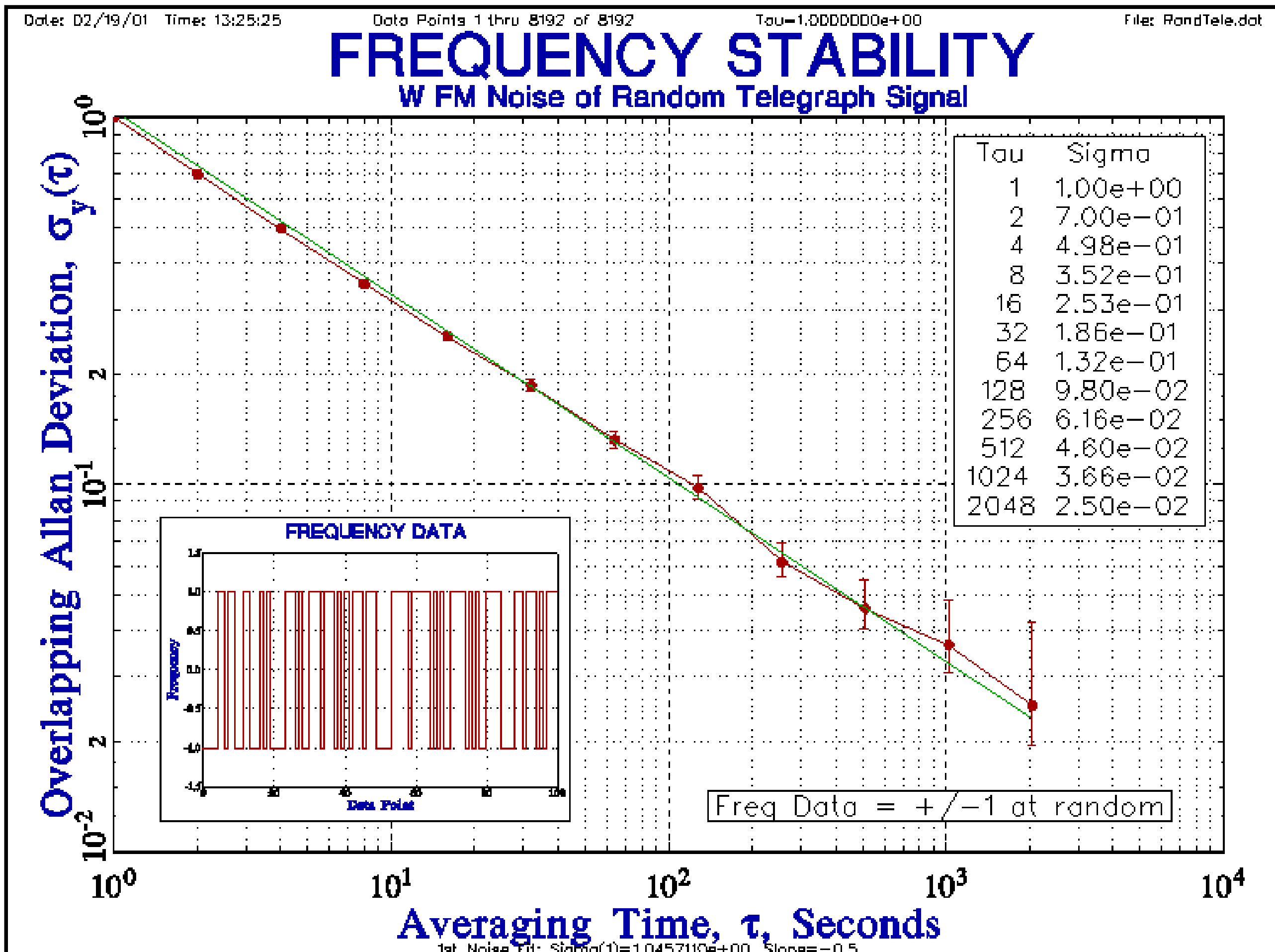


Data Quantization

Symmetricom

While it is desirable to have sufficient resolution that the data is noise rather than resolution limited, highly quantized data can provide good stability information provided that there is a least 1 bit of meaningful variation.

Random telegraph signal representing frequency data having a white FM noise characteristic



Cyclic Environmental Sensitivity

The combination of an environmental sensitivity and a cyclic disturbance will cause the time-domain stability to display a distinctive pattern of maxima and minima at the half period and period of the stimulus as given by the expression:

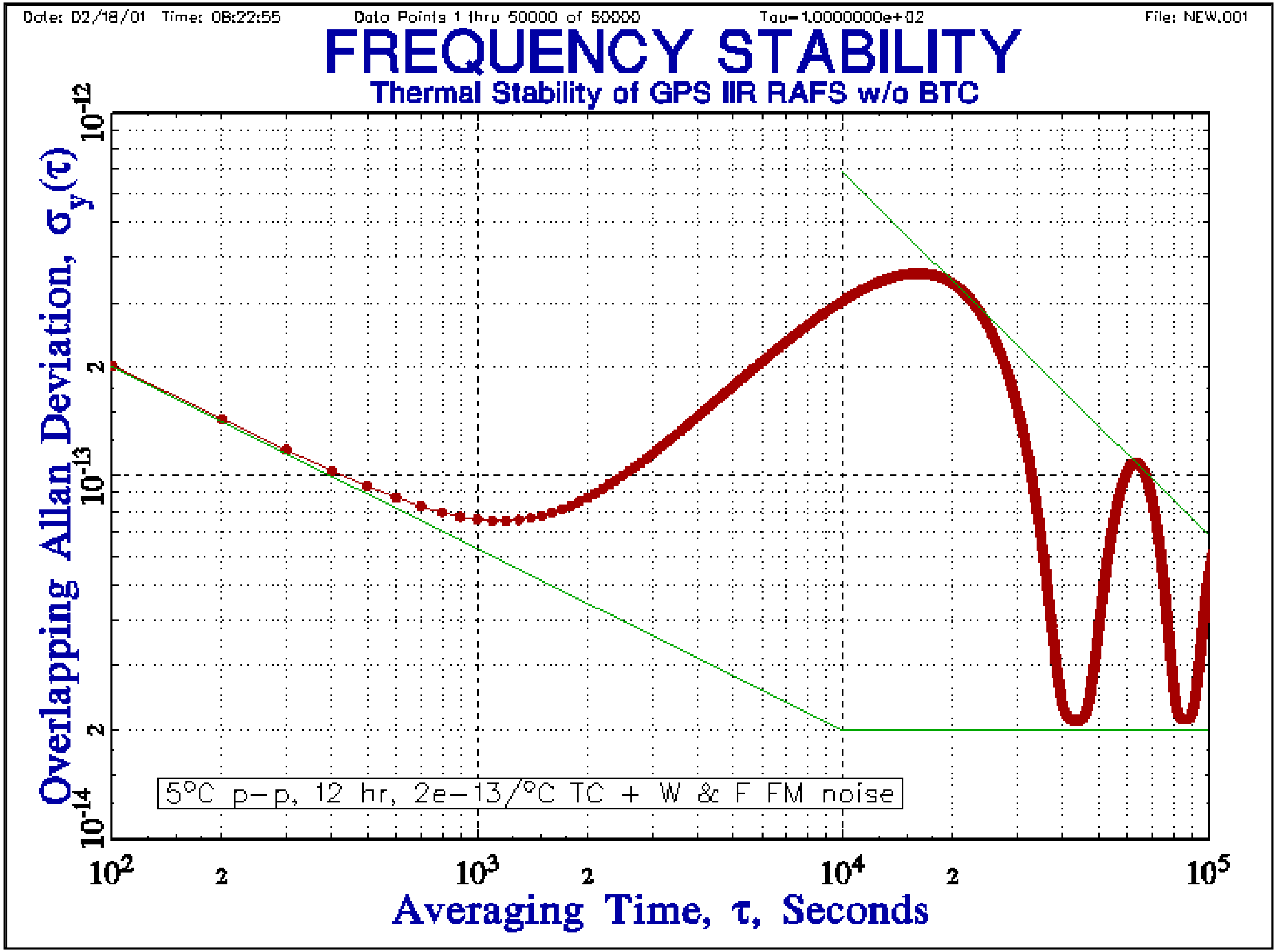
$$\sigma_y(\tau) = \frac{\Delta f / f}{\pi} \cdot \frac{T}{\tau} \sin^2(\pi \cdot \frac{\tau}{T})$$

where: $\Delta f/f$ = peak frequency deviation, T = period of disturbance, and τ = averaging time.

An example of such a stability record is shown below. This is a simulation of the stability of a GPS Block IIR rubidium atomic frequency standard (RAFS) with typical white FM and flicker FM noise levels of $2\text{x}10^{-12}\tau^{-1/2}$ and $2\text{x}10^{-14}$ respectively, plus a temperature coefficient of $2\text{x}10^{-13}/^{\circ}\text{C}$, when exposed to a sinusoidal orbital temperature variation of 5°C p-p with a 12-hour period. The actual clock has a baseplate temperature controller (BTC) that reduces this thermal sensitivity to below the noise level. Even this large periodic variation does not prevent determining the underlying noise level.

Cyclic Environmental Sensitivity

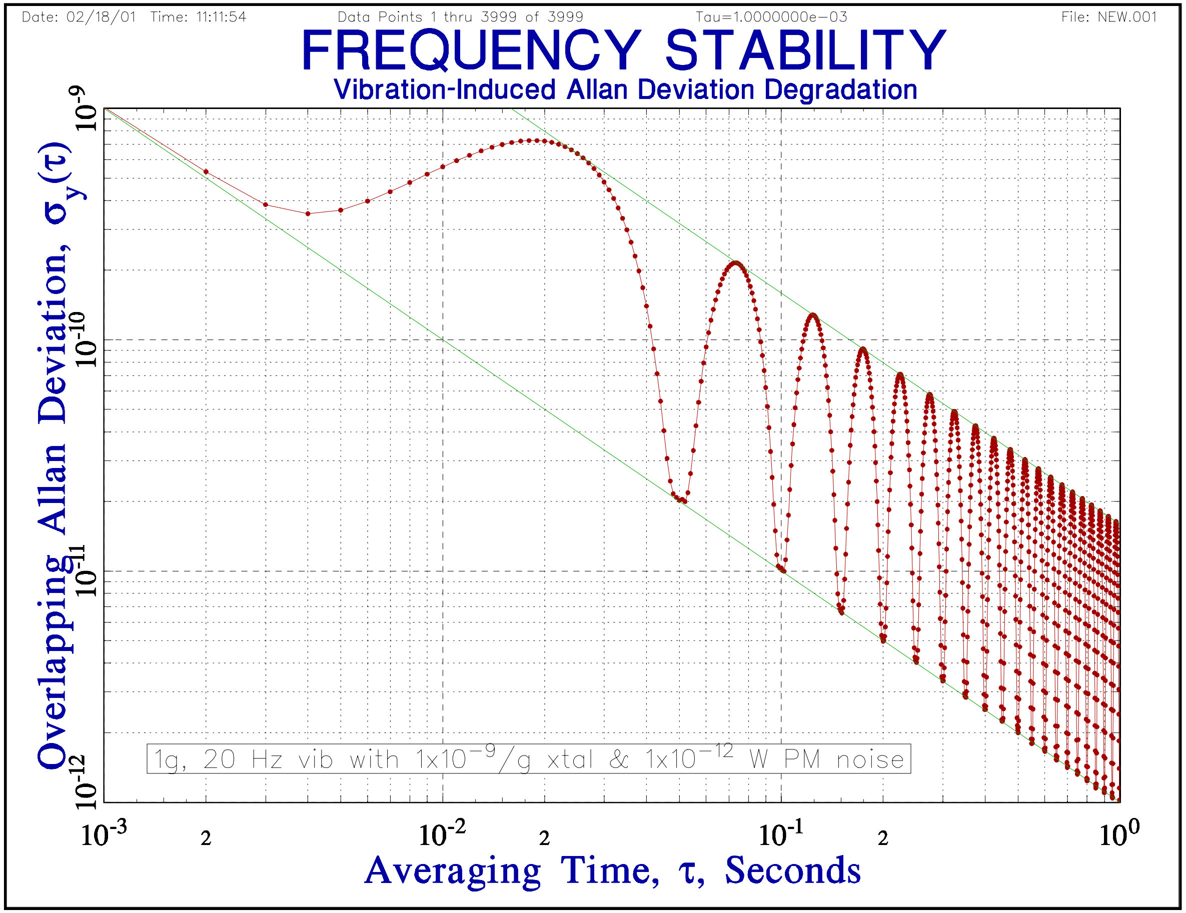
Symmetricom



Vibrational Modulation

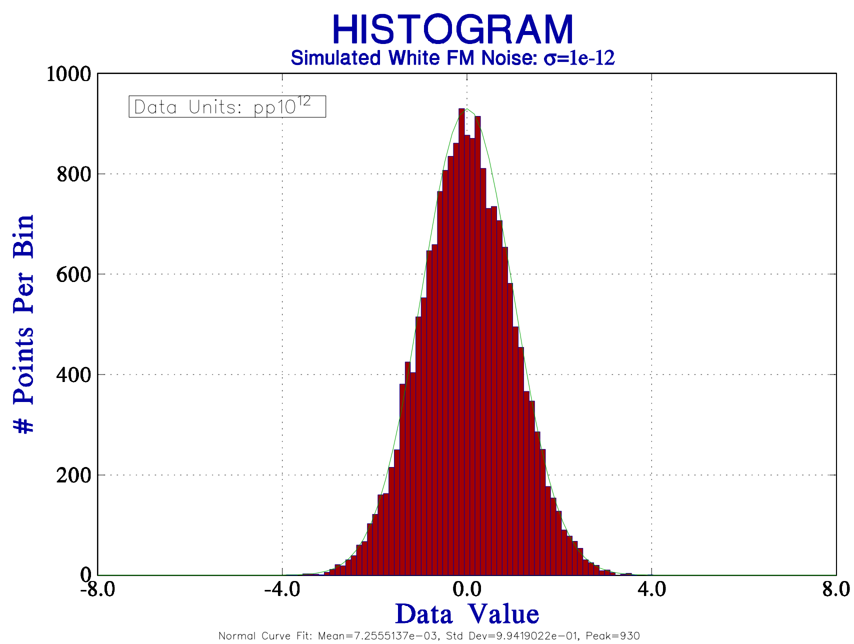
Symmetricom

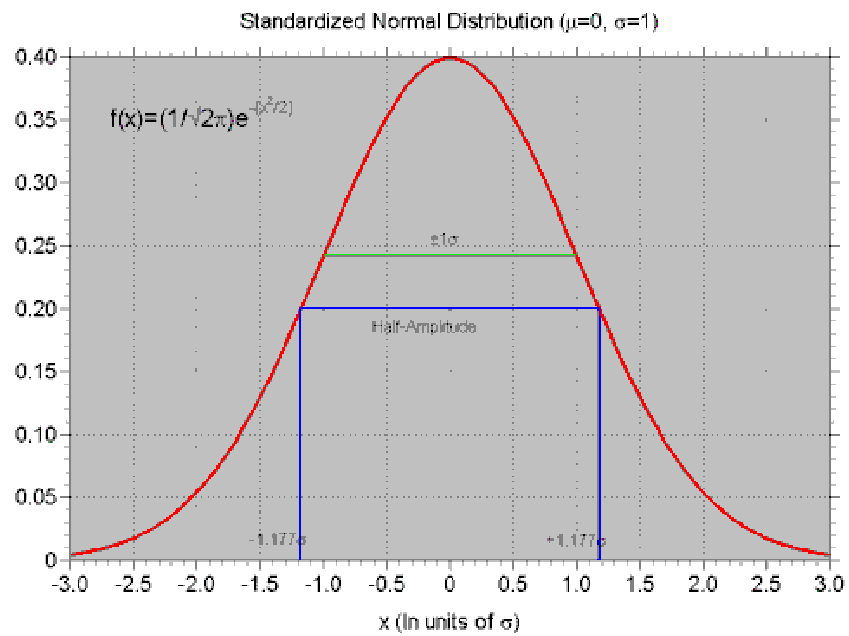
The same general expression applies to the effect of vibrational modulation on the stability of a frequency source. It should be noted that the envelope of the resulting stability plot shows both (at the top) the vibrational FM ($1x10^{-9}\cdot\sqrt{2}$ rms at $1/2\cdot f_{vib}$) and the noise ($1x10^{-12}\tau^{-1}$). Minima occur at averaging times equal to multiples of the vibration period.



Histogram

A histogram shows the amplitude distribution of the phase or frequency fluctuations, and can provide insight regarding them. One can expect a normal (Gaussian) distribution for a reasonably-sized data set, and a different (e.g. bimodal) distribution can be a sign of a problem. For a normal distribution, the standard deviation is approximately equal to the half-width at half-height (HWHA=1.177σ) .





Power Spectral Densities

Frequency stability is described in the frequency domain by several power spectral densities:

PSD of Frequency Fluctuations $S_y(f)$

The power spectral density (PSD) of the fractional frequency fluctuations y(t) in units of 1/Hz is given by $S_y(f) = h(\alpha) \cdot f^{\alpha}$, where f=sideband frequency, Hz and $h(\alpha)$ is an intensity coefficient.

PSD of Phase Fluctuations $S_\phi(f)$

The PSD of the phase fluctuations in units of rad²/Hz is given by $S_\phi(f) = (2\pi\nu_0)^2 \cdot S_x(f)$, where ν_0=carrier frequency, Hz.

PSD of Time Fluctuations $S_x(f)$

The PSD of the time fluctuations x(t) in units of sec²/Hz is given by $S_x(f) = h(\beta) \cdot f^{\beta} = S_y(f)/(2\pi f)^2$, where $\beta=\alpha-2$. The time fluctuations are related to the phase fluctuations by $x(t)= \phi(t)/2\pi\nu_0$.

SSB Phase Noise £(f)

The SSB phase noise in units of dBc/Hz is given by $£(f) = 10 \cdot \log[½ \cdot S_\phi(f)]$. This is the most common engineering unit to specify phase noise.

Domain Conversions

Conversions can be made between time and frequency domain stability measures. These conversions are unique from the frequency domain, but may not be for the opposite case. For the Allan variance, the relationship is:

$$\sigma^2{}_y(\tau) = 2\int_0^{f_h} S_y(f)\frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}df$$

And, for the modified Allan and time deviations, the relationship is:

$$Mod\sigma^2{}_y(\tau = m\tau_0) = \frac{2}{m^4\pi^2\tau^2{}_0}\int_0^{f_h} S_y(f)\frac{\sin^6(\pi\tau f)}{f^2\sin^2(\pi\tau_0 f)}df$$

$$\sigma^2{}_x(\tau = m\tau_0) = \frac{2}{3m^2\pi^2}\int_0^{f_h} S_y(f)\frac{\sin^6(\pi\tau f)}{f^2\sin^2(\pi\tau_0 f)}df$$

These conversions may be performed by numerical integration.

Domain Conversions (Con't)

Domain conversions may be made for power law noise models by using the following conversion formulae:

Noise Type	$\sigma^2_y(\tau)$	$S_y(f)$
RW FM	$A \cdot f^2 \cdot S_y(f) \cdot \tau^1$	$A^{-1} \cdot \tau^{-1} \cdot \sigma^2_y(\tau) \cdot f^{-2}$
F FM	$B \cdot f^1 \cdot S_y(f) \cdot \tau^0$	$B^{-1} \cdot \tau^0 \cdot \sigma^2_y(\tau) \cdot f^{-1}$
W FM	$C \cdot f^0 \cdot S_y(f) \cdot \tau^{-1}$	$C^{-1} \cdot \tau^1 \cdot \sigma^2_y(\tau) \cdot f^0$
F PM	$D \cdot f^{-1} \cdot S_y(f) \cdot \tau^{-2}$	$D^{-1} \cdot \tau^2 \cdot \sigma^2_y(\tau) \cdot f^1$
W PM	$E \cdot f^{-2} \cdot S_y(f) \cdot \tau^{-2}$	$E^{-1} \cdot \tau^2 \cdot \sigma^2_y(\tau) \cdot f^2$

where:

$A = 4\pi^2/6$

$B = 2 \cdot \ln 2$

$C = 1/2$

$D = 1.038 + 3 \cdot \ln(2\pi f_h \tau_0)/4\pi^2$

$E = 3f_h/4\pi^2$

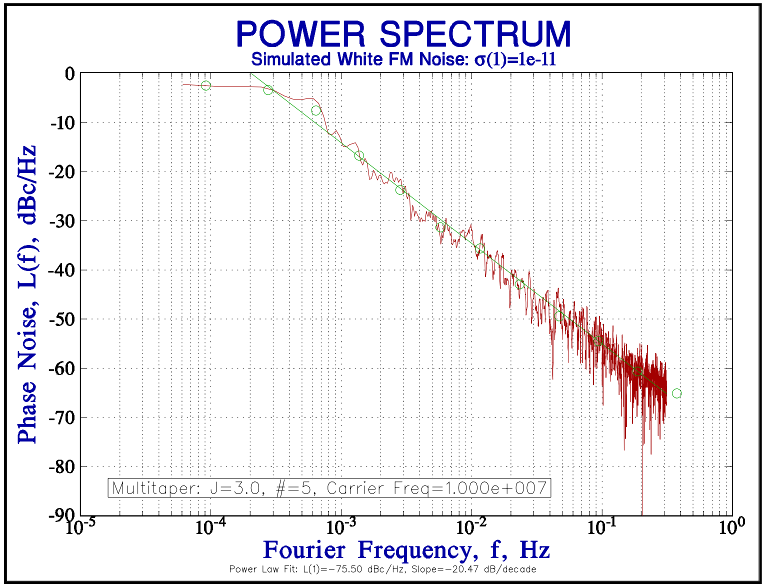
and f_h is the upper cutoff frequency of the measuring system in Hz, and τ_0 is the basic measurement time. The f_h factor applies only to white and flicker PM noise.

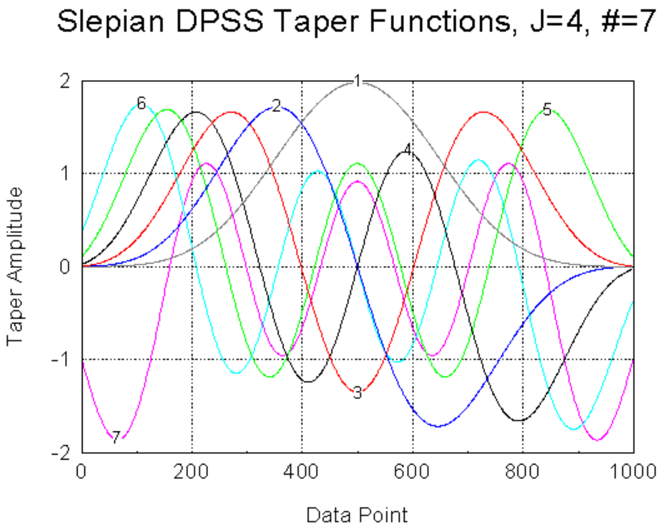
Spectral Analysis

Symmetricom

Spectral analysis is another method to characterize frequency stability. In the context of the time-domain techniques considered here, it can provide additional information about the noise type, and show the presence of interference.

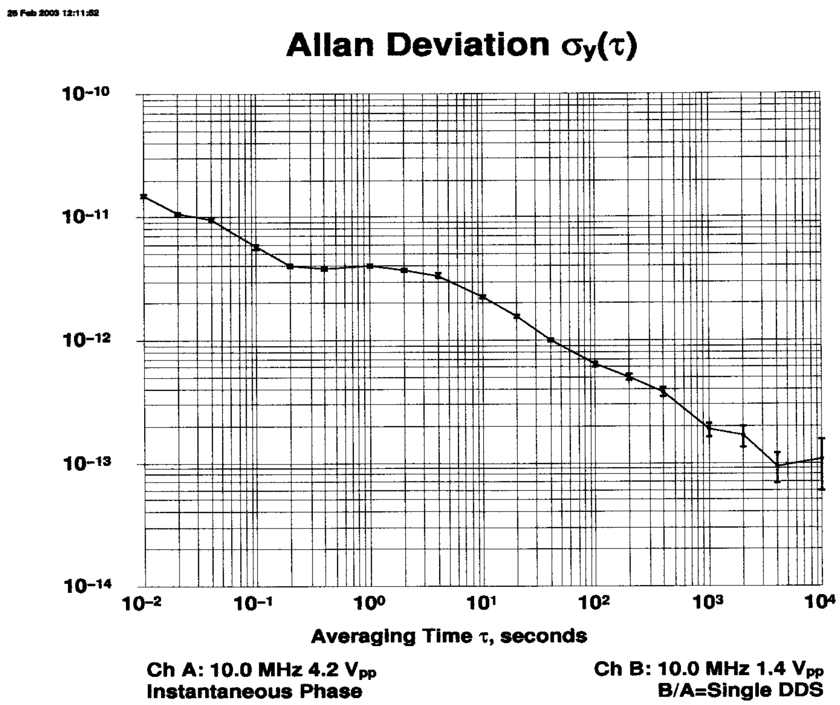
The subject of spectral analysis is a broad one. Issues include reducing bias with windowing (tapering) functions, and reducing variance with averaging (smoothing). A distinguishing aspect of its application to frequency stability analysis is the emphasis on noise, rather than discrete components. Besides FFT-based periodograms, the multitaper method is recommended.

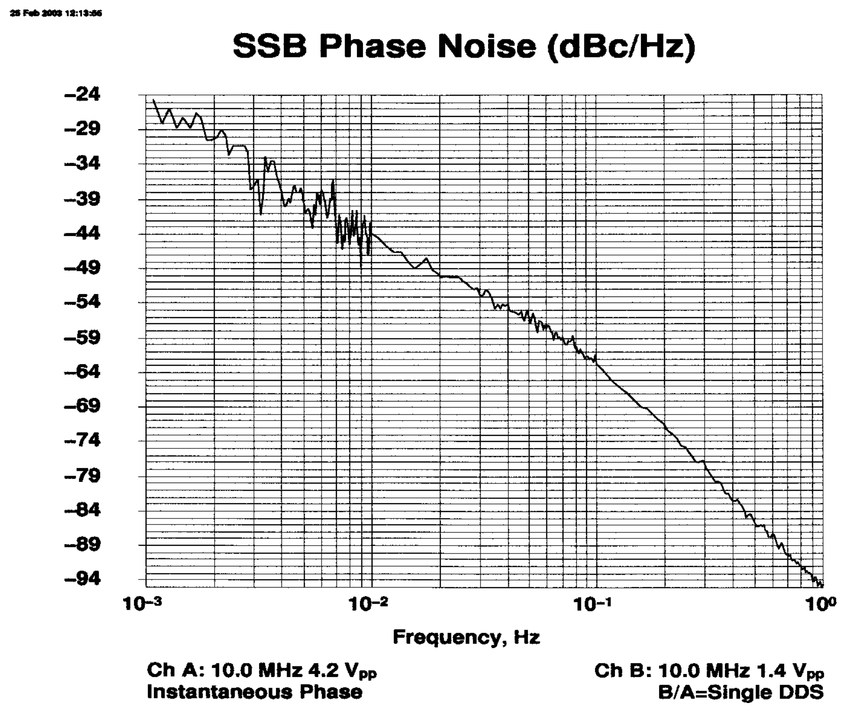




Spectral Analysis (Con't)

Commercial instruments are available [ME-6] to take relatively fast, high resolution time domain phase data and present it both as Allan deviation and £(f), as shown below for a small rubidium frequency standard.





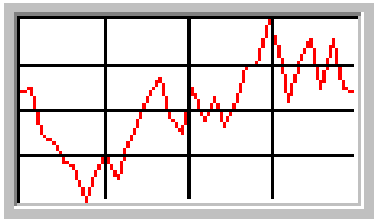
Reporting

The results of a stability analysis are usually presented as a combination of textual, tabular and graphic forms. The text describes the device under test, the test setup, and the methodology of the data preprocessing and analysis, and summarizes the results. The latter often includes a table of the stability statistics. Graphical presentation of the data at each stage of the analysis is generally the most important aspect of presenting the results. For example, these are often a series of plots showing the phase and frequency data with an aging fit, phase and frequency residuals with the aging removed, and stability plots with noise fits and error bars. Plot titles, sub-titles, annotations and inserts can be used to clarify and emphasize the data presentation. The results of several stability runs can be combined, possibly along with specification limits, into a single composite plot. The various elements can be combined into a single electronic document for easy printing and transmittal.

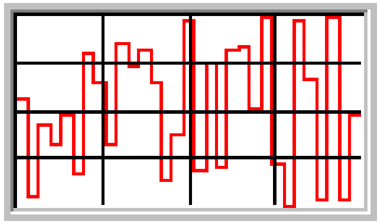
Data Plots

Data plotting is often the most important step in the analysis of frequency stability. Visual inspection can provide vital insight into the results, and is an important "preprocessor" before numerical analysis. A plot also shows much about the validity of a curve fit.

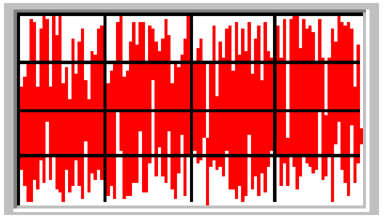
Phase data is generally plotted as line segments connecting the data points. This presentation properly conveys the integral nature of the phase data.



Frequency data is often plotted the same way, simply because that is the way plotting is usually done. But a better presentation is a flat horizontal line between the frequency data points. This shows the averaging time associated with the frequency measurement, and mimics the analog chart record from a counter.



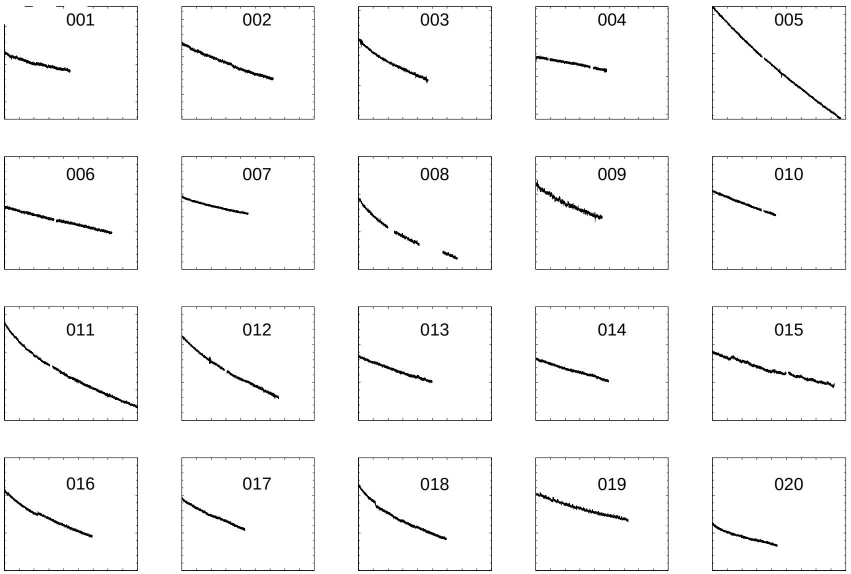
As the density of the data points increases, there is essentially no visual difference between the two plotting methods, and the point method is faster.



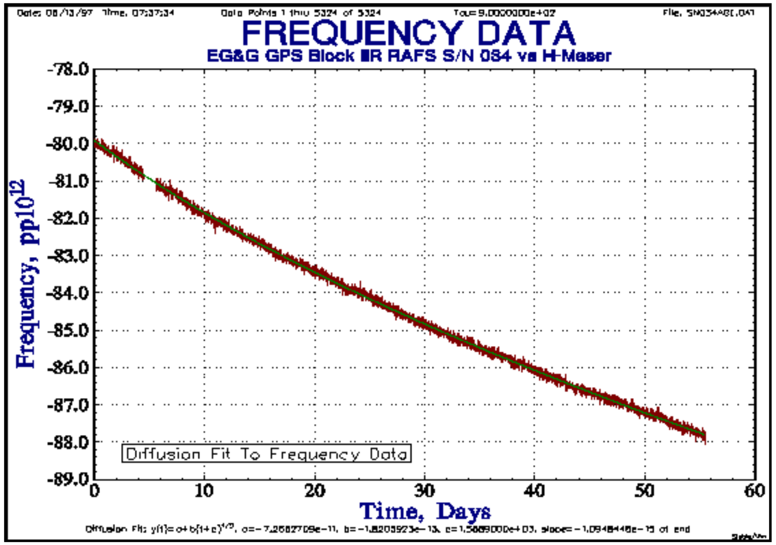
Data Plots (Con’t)

Symmetricom

Small multiple plots can be a useful visual tool for comparing behavior, as shown in these aging plots for GPS Rb clocks [SW-7].



Fits to linear, log, diffusion or other drift models can support stability analysis and aid in understanding the physical processes involved.



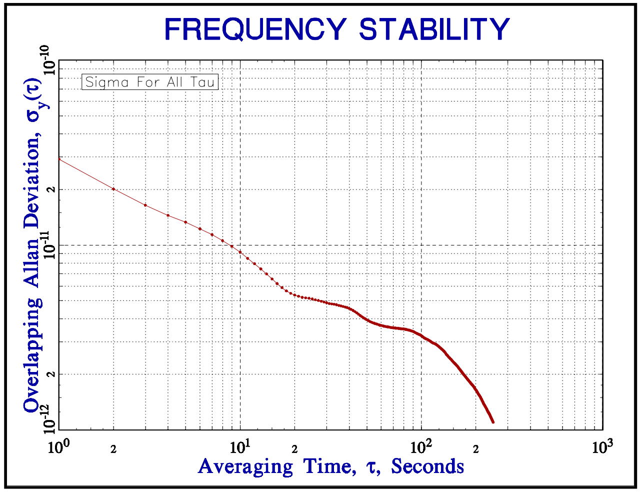
Stability Plots

Stability plots generally take the form of graphs of log versus log , often with error bars to shown the precision of the results. The slope of the $\sigma_y(\tau)$ characteristic depends on the type of noise. It is customary to show points at binary increments of tau. These are equally spaced on the log scale, and are the result of successive averaging by two. Such a run usually ends when there are too few analysis points (say < 7) for reasonable confidence. A run for all possible values, while slow, can provide valuable information since it is, in effect, a form of spectral analysis that can show periodic instabilities such as environmental effects. Such an all-tau run can be made much faster by spacing the points no closer than can be seen on the display device.

All Tau

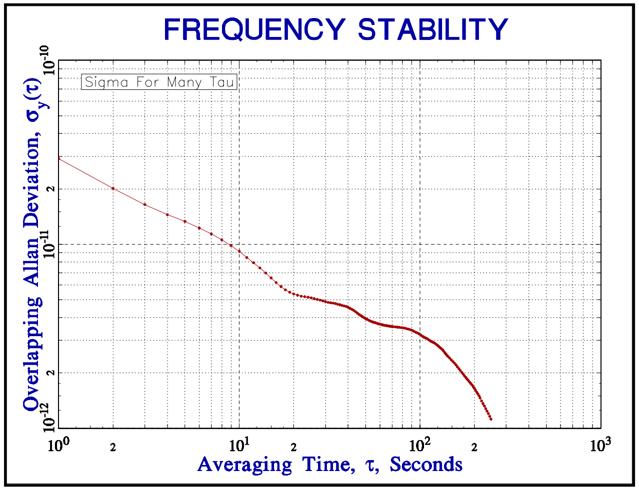
Stability calculations made at all possible tau values can provide an excellent indication of the variations in the results, and are a simple form of spectral analysis. In particular, cyclic variations are often the result of interference between the sampling rate and some periodic instability (such as environmental sensitivity).

An all tau analysis is computationally-intensive and can therefore be slow. For most purposes, however, it is not necessary to calculate values at every tau, but instead to do so at enough points to provide a nearly continuous curve on the display device (screen or paper). Such a “many tau” analysis can be orders-of-magnitude faster and yet provide the same information.



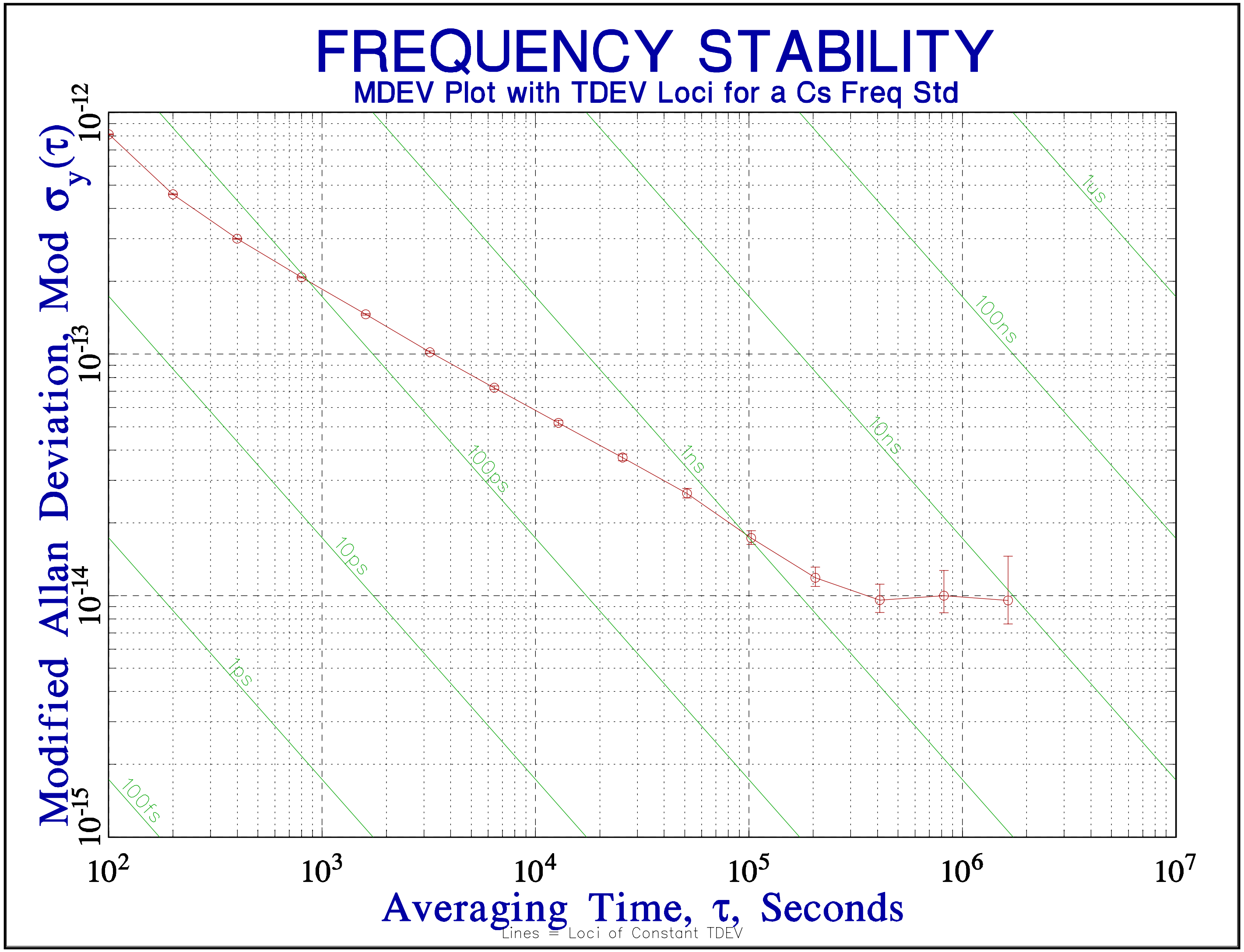
Many tau calculates just enough points to produce a smooth plot

➔



MDEV and TDEV

Loci of constant TDEV can be added to an MDEV plot



Software Validation

Several methods are available to validate frequency stability analysis software:

1. Manual Analysis: The results obtained by manual analysis of small data sets (such as NBS Monograph 140 Annex 8.E [G-4]) can be compared with the new program output. This is always good to do to get a "feel" for the process.
2. Published Results: The results of a published analysis or test suite can be compared with the new program output [SW-3], [SW-4].
3. Other Programs: The results obtained from other specialized stability analysis programs [SW-6], or from a previous generation computer or operating system, can be compared with the new program output.
4. General Programs: The results obtained from industry standard, general purpose mathematical and spreadsheet programs (such as MathCAD and Excel) can be compared with the new program output.

Software Validation (Con't)

5. Consistency Checks: The new program should be verified for internal consistency, such as producing the same stability results from phase and frequency data. The standard and Allan variances should be approximately equal for white FM noise. The Allan and modified Allan variances, and total variance, should be identical for an averaging factor of 1. For other averaging factors, the modified Allan variance should be approximately one-half the normal Allan variance for white FM noise and $\tau >> \tau_0$. The normal and overlapping Allan variances, and total variance, should be approximately equal, while the overlapping method and total variance provide better confidence of the stability estimates. The various methods of drift removal should yield similar results.
6. Simulated Data: Simulated clock data can also serve as a useful cross check. Known values of frequency offset and drift can be inserted, analyzed and removed. Known values of power law noise can be generated, analyzed, plotted and modeled.

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